Some Applications of the Representation Theory of Equipped Posets.

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30/11/2018
Talk I

Equipped Posets and Their Category of Representations. 9/11/2018.

Talk II


Talk III

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Equipped Posets and Their Category of Representations.

Talk II

Some Algorithms of Differentiation to Classify some Equipped Posets.
16/11/2018.

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Road Map

1. G.E. Andrews’s problems and Equipped posets
   - Delannoy numbers

2. Auslander-Reiten quiver from a TDA
   - Categories of type \( \text{add } Y_t \)

3. Digital Watermarking

4. References
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The Main Goal
Aims and Scope

In this talk, we define some ordered integer partitions whose structure is associated to some equipped posets. Also, the relationship of Auslander-Reiten quiver and TDA, and Digital Watermarking using *DVII*. 
Remember that

The category of representations of an equipped poset over a pair of fields \((F, G)\) (where \(G\) is a quadratic extension of \(F\)) is defined as a system of the form

\[ U = (U_0 ; U_x \mid x \in \mathcal{P}), \tag{1} \]

where \(U_0\) is a finite dimensional \(F\)-space and for each \(x \in \mathcal{P}, U_x\) is a \(G\)-subspace of \(\widetilde{U}_0\), such that,

\[ x \leq y \implies U_x \subset U_y \]

\[ x \preceq y \implies F(U_x) \subset U_y. \]
Remember that

The category of representations of an equipped poset over a pair of fields $(F, G)$ (where $G$ is a quadratic extension of $F$) is defined as a system of the form

$$U = \left( U_0 ; \ U_x \mid x \in \mathcal{P} \right),$$

(1)

where $U_0$ is a finite dimensional $F$-space and for each $x \in \mathcal{P}$, $U_x$ is a $G$-subspace of $\tilde{U}_0$, such that,

$$x \leq y \implies U_x \subset U_y$$

$$x \trianglelefteq y \implies F(U_x) \subset U_y.$$
For each \( x \in \mathcal{P} \) we let \( U_x \) denote the radical subspace of \( U_x \) such that
\[
U_x = \sum_{z \triangleleft x} F(U_z) + \sum_{z \preceq x} U_z.
\]

**Dimension**

The *dimension* of a representation \( U \in \text{rep} \mathcal{P} \) is a vector \( d \) such that
\[
d = \text{dim} \ U = (d_0; d_x \mid x \in \mathcal{P})
\]
where \( d_0 = \text{dim}_F U_0 \) and \( d_x = \text{dim}_G U_x/U_x \).
For each $x \in \mathcal{P}$ we let $U_x$ denote the radical subspace of $U_x$ such that

$$U_x = \sum_{z \prec x} F(U_z) + \sum_{z \prec x} U_z.$$ 

**Dimension**

The *dimension* of a representation $U \in \text{rep } \mathcal{P}$ is a vector $d$ such that

$$d = \dim U = (d_0; d_x \mid x \in \mathcal{P})$$

where $d_0 = \dim_F U_0$ and $d_x = \dim_G U_x / \overline{U_x}$. 
Integer Partitions
**Definition**

A *partition* \( \lambda \) of a positive integer \( n \) is a nonincreasing sequence of positive integers \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) such that

\[
n = \sum_{i=1}^{n} \lambda_i
\]

A *composition* is a partition for which the order of its parts matters.
Example

\{1, 1, 1\}, \{2, 1\}, and \{3\} are the three partitions of 3, whereas
\{1, 1, 1\}, \{2, 1\}, \{1, 2\} and \{3\} are the four compositions of 3.
Regarding partitions and compositions there are numerous open problems. For instance in 1987 G.E. Andrews proposed the following problems:

1. For what sets of positive integers $S$ and $T$ is $P(S, n) = P(T, n - a)$ for $n \geq a$ with $a$ fixed.

2. For each pair $S$ and $T$ which answer question (1) can a bijection be found between the partitions of $n$ into elements of $S$ and the partitions of $n - a$ into elements of $T$?
For $a = 1$, some identities introduced by Gessel and Stanton imply the solutions:

\[ S = \{ n \mid n \text{ odd or } n \equiv \pm 2, \pm 8, \pm 12, \pm 14 \pmod{32} \}, \]
\[ T = \{ n \mid n \text{ odd or } n \equiv \pm 2, \pm 8, \pm 12, \pm 14 \pmod{32} \}. \]
\[ S = \{ n \mid n \equiv \pm 1, \pm 4, \pm 5, \pm 6, \pm 7, \pm 9, \pm 10, \pm 11, \pm 13, \pm 15, \pm 16, 19 \} \]
\[ T = \{ n \mid n \equiv \pm 1, \pm 3, \pm 4, \pm 5, \pm 9, \pm 10, \pm 11, \pm 14, \pm 15, \pm 16, \pm 17, \pm 19 \}. \]

all of them \( \text{ mod } 40. \)
The problem is still open if we consider integer compositions.
Definition

Let \((D, \sqsubseteq)\) be a partially ordered set of integer compositions \(\{x_1, x_2, x_3, x_4\}\) such that:

1. \(x_i \geq 0, \ 1 \leq i \leq 4\),
2. At least two of its elements are positive,
3. \(x_2 = x_4\), and the difference \(x_3 - x_1 \geq 0\).

Besides, \(\{x_1, x_2, x_3, x_4\} \sqsubseteq \{x'_1, x'_2, x'_3, x'_4\}\) if and only if \(x'_1 \leq x_1\), \(x'_3 \leq x_3\), \(x_2 \leq x'_2\) and \(x_4 \leq x'_4\).
It is clear that if $\mathcal{D}_n$ denotes the set of compositions of type $\mathcal{D}$ of a fixed integer $n \geq 2$ then:

$$\mathcal{D} = \bigcup_{n \geq 2} \mathcal{D}_n$$
Theorem

The poset of compositions $D_n$ of type $D$ of a fixed integer $n \geq 2$ is a sum of $\left\lfloor \frac{n}{2} \right\rfloor$ chains.
Proof. The set of minimal points of $D_n$ consists of all compositions $\{x, y, z, w\}$ such that $y = w = 0$, $x + z = n$. Thus $x \in \{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \}$. □
Composition of the number $n = 4$

\{0, 0, 0, 4\} \{0, 0, 4, 0\} \{0, 4, 0, 0\} \{4, 0, 0, 0\}

\{0, 0, 3, 1\} \{0, 3, 0, 1\} \{0, 3, 1, 0\} \{3, 0, 1, 0\} \{3, 0, 0, 1\} \{3, 1, 0, 0\}

\{0, 0, 1, 3\} \{0, 1, 0, 3\} \{0, 1, 3, 0\} \{1, 0, 3, 0\} \{1, 0, 0, 3\} \{1, 3, 0, 0\}

\{0, 0, 2, 2\} \{0, 2, 0, 2\} \{0, 2, 2, 0\} \{2, 0, 2, 0\} \{2, 0, 0, 2\} \{2, 2, 0, 0\}

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\{1, 1, 1, 1\}
Composition of the number $n = 4$

\[
\begin{align*}
\{0, 0, 0, 4\} & \{0, 0, 4, 0\} & \{0, 4, 0, 0\} & \{4, 0, 0, 0\} \\
\{0, 0, 3, 1\} & \{0, 3, 0, 1\} & \{0, 3, 1, 0\} & \{3, 0, 1, 0\} & \{3, 0, 0, 1\} & \{3, 1, 0, 0\} \\
\{0, 0, 1, 3\} & \{0, 1, 0, 3\} & \{0, 1, 3, 0\} & \{1, 0, 3, 0\} & \{1, 0, 0, 3\} & \{1, 3, 0, 0\} \\
\{0, 0, 2, 2\} & \{0, 2, 0, 2\} & \{0, 2, 2, 0\} & \{2, 0, 2, 0\} & \{2, 0, 0, 2\} & \{2, 2, 0, 0\} \\
\{0, 1, 1, 2\} & \{1, 0, 1, 2\} & \{1, 1, 0, 2\} & \{1, 1, 2, 0\} \\
\{0, 1, 2, 1\} & \{1, 0, 2, 1\} & \{1, 2, 0, 1\} & \{1, 2, 1, 0\} \\
\{0, 2, 1, 1\} & \{2, 0, 1, 1\} & \{2, 1, 0, 1\} & \{2, 1, 1, 0\} \\
\{1, 1, 1, 1\}
\end{align*}
\]
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\{0,0,1,3\} & \quad \{0,1,0,3\} & \quad \{0,1,3,0\} & \quad \{1,0,3,0\} & \quad \{1,0,0,3\} & \quad \{1,3,0,0\} \\
\{0,0,2,2\} & \quad \{0,2,0,2\} & \quad \{0,2,2,0\} & \quad \{2,0,2,0\} & \quad \{2,0,0,2\} & \quad \{2,2,0,0\} \\
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\{0,2,1,1\} & \quad \{2,0,1,1\} & \quad \{2,1,0,1\} & \quad \{2,1,1,0\} \\
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\end{align*}
\]
Composition of the number $n = 4$

$$\{0,0,0,4\} \{0,0,4,0\} \{0,4,0,0\} \{4,0,0,0\}$$
$$\{0,0,3,1\} \{0,3,0,1\} \{0,3,1,0\} \{3,0,1,0\} \{3,0,0,1\} \{3,1,0,0\}$$
$$\{0,0,1,3\} \{0,1,0,3\} \{0,1,3,0\} \{1,0,3,0\} \{1,0,0,3\} \{1,3,0,0\}$$
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Composition of the number \( n = 4 \)

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\begin{align*}
\{0, 0, 0, 4\} & \quad \{0, 0, 4, 0\} & \quad \{0, 4, 0, 0\} & \quad \{4, 0, 0, 0\} \\
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\{0, 0, 1, 3\} & \quad \{0, 1, 0, 3\} & \quad \{0, 1, 3, 0\} & \quad \{1, 0, 3, 0\} & \quad \{1, 0, 0, 3\} & \quad \{1, 3, 0, 0\} \\
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\{0, 1, 1, 2\} & \quad \{1, 0, 1, 2\} & \quad \{1, 1, 0, 2\} & \quad \{1, 1, 2, 0\} \\
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\{1, 1, 1, 1\}
\end{align*}
\]
### Composition of the number \( n = 4 \) of type \( D \)

\[
\begin{align*}
\{0, 0, 0, 4\} & \quad \{0, 0, 4, 0\} & \quad \{0, 4, 0, 0\} & \quad \{4, 0, 0, 0\} \\
\{0, 0, 3, 1\} & \quad \{0, 3, 0, 1\} & \quad \{0, 3, 1, 0\} & \quad \{3, 0, 1, 0\} & \quad \{3, 0, 0, 1\} & \quad \{3, 1, 0, 0\} \\
\{0, 0, 1, 3\} & \quad \{0, 1, 0, 3\} & \quad \{0, 1, 3, 0\} & \quad \{1, 0, 3, 0\} & \quad \{1, 0, 0, 3\} & \quad \{1, 3, 0, 0\} \\
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\{0, 1, 1, 2\} & \quad \{1, 0, 1, 2\} & \quad \{1, 1, 0, 2\} & \quad \{1, 1, 2, 0\} \\
\{0, 1, 2, 1\} & \quad \{1, 0, 2, 1\} & \quad \{1, 2, 0, 1\} & \quad \{1, 2, 1, 0\} \\
\{0, 2, 1, 1\} & \quad \{2, 0, 1, 1\} & \quad \{2, 1, 0, 1\} & \quad \{2, 1, 1, 0\} \\
\{1, 1, 1, 1\}
\end{align*}
\]
Composition of the number $n = 4$ of type $\mathcal{D}$

$$\mathcal{D}_4 = \{\{1, 0, 3, 0\}, \{0, 2, 0, 2\}, \{2, 0, 2, 0\}, \{0, 1, 2, 1\}, \{1, 1, 1, 1\}\}$$
Composition of the number \( n = 4 \) of type \( \mathcal{D} \)

\[
\mathcal{D}_4 = \{\{1, 0, 3, 0\}, \{0, 2, 0, 2\}, \{2, 0, 2, 0\}, \{0, 1, 2, 1\}, \{1, 1, 1, 1\}\}
\]
### Composition of the number $n = 5$

$\{0, 0, 0, 5\}$ $\{0, 0, 5, 0\}$ $\{0, 5, 0, 0\}$ $\{5, 0, 0, 0\}$

$\{0, 0, 4, 1\}$ $\{0, 4, 0, 1\}$ $\{0, 4, 1, 0\}$ $\{4, 0, 1, 0\}$ $\{4, 0, 0, 1\}$ $\{4, 1, 0, 0\}$

$\{0, 0, 1, 4\}$ $\{0, 1, 0, 4\}$ $\{0, 1, 4, 0\}$ $\{1, 0, 4, 0\}$ $\{1, 0, 0, 4\}$ $\{1, 4, 0, 0\}$

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\end{align*}
\]
### Composition of the number $n = 5$

- $\{0, 0, 0, 5\}$  $\{0, 0, 5, 0\}$  $\{0, 5, 0, 0\}$  $\{5, 0, 0, 0\}$
- $\{0, 0, 4, 1\}$  $\{0, 4, 0, 1\}$  $\{0, 4, 1, 0\}$  $\{4, 0, 1, 0\}$  $\{4, 0, 0, 1\}$  $\{4, 1, 0, 0\}$
- $\{0, 0, 1, 4\}$  $\{0, 1, 0, 4\}$  $\{0, 1, 4, 0\}$  $\{1, 0, 4, 0\}$  $\{1, 0, 0, 4\}$  $\{1, 4, 0, 0\}$
- $\{0, 0, 3, 2\}$  $\{0, 3, 0, 2\}$  $\{0, 3, 2, 0\}$  $\{3, 0, 2, 0\}$  $\{3, 0, 0, 2\}$  $\{3, 2, 0, 0\}$
- $\{0, 0, 2, 3\}$  $\{0, 2, 0, 3\}$  $\{0, 2, 3, 0\}$  $\{2, 0, 3, 0\}$  $\{2, 0, 0, 3\}$  $\{2, 3, 0, 0\}$
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- $\{1, 3, 0, 1\}$  $\{1, 3, 1, 0\}$  $\{0, 3, 1, 1\}$  $\{3, 0, 1, 1\}$  $\{3, 1, 0, 1\}$  $\{3, 1, 1, 0\}$
- $\{0, 1, 2, 2\}$  $\{1, 0, 2, 2\}$  $\{1, 2, 0, 2\}$  $\{1, 2, 2, 0\}$  $\{0, 2, 1, 2\}$  $\{2, 0, 1, 2\}$
- $\{2, 1, 0, 2\}$  $\{2, 1, 2, 0\}$  $\{0, 2, 2, 1\}$  $\{2, 0, 2, 1\}$  $\{2, 2, 0, 1\}$  $\{2, 2, 1, 0\}$
- $\{1, 1, 1, 2\}$  $\{1, 1, 2, 1\}$  $\{1, 2, 1, 1\}$  $\{2, 1, 1, 1\}$
### Composition of the number $n = 5$

```
{0, 0, 0, 5} {0, 0, 5, 0} {0, 5, 0, 0} {5, 0, 0, 0} 
{0, 0, 4, 1} {0, 4, 0, 1} {0, 4, 1, 0} {4, 0, 1, 0} {4, 0, 0, 1} {4, 1, 0, 0} 
{0, 0, 1, 4} {0, 1, 0, 4} {0, 1, 4, 0} {1, 0, 4, 0} {1, 0, 0, 4} {1, 4, 0, 0} 
{0, 0, 3, 2} {0, 3, 0, 2} {0, 3, 2, 0} {3, 0, 2, 0} {3, 0, 0, 2} {3, 2, 0, 0} 
{0, 0, 2, 3} {0, 2, 0, 3} {0, 2, 3, 0} {2, 0, 3, 0} {2, 0, 0, 3} {2, 3, 0, 0} 
{0, 1, 1, 3} {1, 0, 1, 3} {1, 1, 0, 3} {1, 1, 3, 0} {0, 1, 3, 1} {1, 0, 3, 1} 
{1, 3, 0, 1} {1, 3, 1, 0} {0, 3, 1, 1} {3, 0, 1, 1} {3, 1, 0, 1} {3, 1, 1, 0} 
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{2, 1, 0, 2} {2, 1, 2, 0} {0, 2, 2, 1} {2, 0, 2, 1} {2, 2, 0, 1} {2, 2, 1, 0} 
{1, 1, 1, 2} {1, 1, 2, 1} {1, 2, 1, 1} {2, 1, 1, 1} 
```

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Composition of the number \( n = 5 \)

\[
\begin{align*}
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&\{0,0,4,1\} \quad \{0,4,0,1\} \quad \{0,4,1,0\} \quad \{4,0,1,0\} \quad \{4,0,0,1\} \quad \{4,1,0,0\} \\
&\{0,0,1,4\} \quad \{0,1,0,4\} \quad \{0,1,4,0\} \quad \{1,0,4,0\} \quad \{1,0,0,4\} \quad \{1,4,0,0\} \\
&\{0,0,3,2\} \quad \{0,3,0,2\} \quad \{0,3,2,0\} \quad \{3,0,2,0\} \quad \{3,0,0,2\} \quad \{3,2,0,0\} \\
&\{0,0,2,3\} \quad \{0,2,0,3\} \quad \{0,2,3,0\} \quad \{2,0,3,0\} \quad \{2,0,0,3\} \quad \{2,3,0,0\} \\
&\{0,1,1,3\} \quad \{1,0,1,3\} \quad \{1,1,0,3\} \quad \{1,1,3,0\} \quad \{0,1,3,1\} \quad \{1,0,3,1\} \\
&\{1,3,0,1\} \quad \{1,3,1,0\} \quad \{0,3,1,1\} \quad \{3,0,1,1\} \quad \{3,1,0,1\} \quad \{3,1,1,0\} \\
&\{0,1,2,2\} \quad \{1,0,2,2\} \quad \{1,2,0,2\} \quad \{1,2,2,0\} \quad \{0,2,1,2\} \quad \{2,0,1,2\} \\
&\{2,1,0,2\} \quad \{2,1,2,0\} \quad \{0,2,2,1\} \quad \{2,0,2,1\} \quad \{2,2,0,1\} \quad \{2,2,1,0\} \\
&\{1,1,1,2\} \quad \{1,1,2,1\} \quad \{1,2,1,1\} \quad \{2,1,1,1\}
\end{align*}
\]
Composition of the number $n = 5$ of type $\mathcal{D}$

\[
\begin{align*}
\{0,0,0,5\} & \{0,0,5,0\} \{0,5,0,0\} \{5,0,0,0\} \\
\{0,0,4,1\} & \{0,4,0,1\} \{0,4,1,0\} \{4,0,1,0\} \{4,0,0,1\} \{4,1,0,0\} \\
\{0,0,1,4\} & \{0,1,0,4\} \{0,1,4,0\} \{1,0,4,0\} \{1,0,0,4\} \{1,4,0,0\} \\
\{0,0,3,2\} & \{0,3,0,2\} \{0,3,2,0\} \{3,0,2,0\} \{3,0,0,2\} \{3,2,0,0\} \\
\{0,0,2,3\} & \{0,2,0,3\} \{0,2,3,0\} \{2,0,3,0\} \{2,0,0,3\} \{2,3,0,0\} \\
\{0,1,1,3\} & \{1,0,1,3\} \{1,1,0,3\} \{1,1,3,0\} \{0,1,3,1\} \{1,0,3,1\} \\
\{1,3,0,1\} & \{1,3,1,0\} \{0,3,1,1\} \{3,0,1,1\} \{3,1,0,1\} \{3,1,1,0\} \\
\{0,1,2,2\} & \{1,0,2,2\} \{1,2,0,2\} \{1,2,2,0\} \{0,2,1,2\} \{2,0,1,2\} \\
\{2,1,0,2\} & \{2,1,2,0\} \{0,2,2,1\} \{2,0,2,1\} \{2,2,0,1\} \{2,2,1,0\} \\
\{1,1,1,2\} & \{1,1,2,1\} \{1,2,1,1\} \{2,1,1,1\}
\end{align*}
\]
Composition of the number $n = 5$ of type $\mathcal{D}$

$$\mathcal{D}_5 = \{\{1, 0, 4, 0\}, \{2, 0, 3, 0\}, \{0, 1, 3, 1\}, \{0, 2, 1, 2\}, \{1, 1, 2, 1\}\}$$
Composition of the number \( n = 5 \) of type \( \mathcal{D} \)

\[
\mathcal{D}_5 = \{\{1, 0, 4, 0\}, \{2, 0, 3, 0\}, \{0, 1, 3, 1\}, \{0, 2, 1, 2\}, \{1, 1, 2, 1\}\}
\]
Some Applications of the Representation Theory of Equipped Posets.

Compositions in $\mathcal{D}_4$ and $\mathcal{D}_5$

$\mathcal{D}_4$

- $\{0,1,2,1\}$
  - $\{0,2,0,2\}$
  - $\{1,0,3,0\}$

$\mathcal{D}_5$

- $\{0,1,3,1\}$
  - $\{0,2,1,2\}$
  - $\{1,0,4,0\}$

$\{0,2,1,2\}$

- $\{1,1,1,1\}$
  - $\{2,0,2,0\}$
  - $\{1,1,2,1\}$
Example

\{2, 0, 4, 0\} \trianglelefteq \{1, 1, 3, 1\} \trianglelefteq \{0, 2, 2, 2\} is a chain of compositions of type $\mathcal{D}$ of 6.
Some Applications of the Representation Theory of Equipped Posets.
Regarding antichains in $\mathcal{D}_n$ we have the following result:
The number $D_n^2$ of two-points antichains contained in $D_n$ is given by the formula:

$$n+1\left\lceil \frac{i}{2} \right\rceil - 1 \sum_{i=2}^{n+1} \sum_{j=0}^{\left\lceil \frac{i}{2} \right\rceil - 1} h_{ij}(t_i - 2t_j)$$

where

$$h_{ij} = \begin{cases} 
0, & \text{if } i = n + 1 \text{ and } j = 0, \\
n - i + 2, & \text{if } j = \left\lceil \frac{i}{2} \right\rceil - 1 > 0, \\
1, & \text{otherwise.}
\end{cases}$$

$$D_n^2 = \{2, 10, 29, 66, 129, 228, 374, \ldots \}.$$
The structure of posets $D_n$ allows us to give the following result regarding the Andrews’s problems:
Corollary

Let $C(n, \mathcal{D})$ be the number of compositions of type $\mathcal{D}$ of the positive integer $n$ then $C(2n + 1, \mathcal{D}) = C(2n, \mathcal{D}), \quad n \geq 1$. 
Proof. Since for each $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, there are $i + 1$ compositions \{x, y, z, y\} of type $D$ with $x + y = i$ then

$$C(2n + 1, D) = t_{\lfloor n + \frac{1}{2} \rfloor + 1} - 1 = C(2n, D).$$
<table>
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<th>Delannoy numbers</th>
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**Delannoy Numbers**
Our Research

*The enumeration of lattice paths is an extensively developed subject. The point in this work is that certain lattice path problems are equivalent to determining the number of some restricted compositions of a positive integer n.*
Delannoy numbers
Numbers $D(x_0, k_0)$ are called Delannoy numbers $d(x, y)$ which can be obtained by counting the number of lattice paths in $\mathbb{N}^2$ from $(0,0)$ to $(x,y)$ considering directions $(1,0)$, $(0,1)$ and $(1,1)$. 
Delannoy numbers

1 1 1 1 1 1 1
1 13 85 377 1289 3653 8989
1 11 61 231 681 1683 3653
1 9 41 129 321 681 1289
1 7 25 63 129 231 377
1 5 13 25 41 61 85
1 3 5 7 9 11 13
1 1 1 1 1 1 1
We associate to each composition \( \{x_1, x_2, x_3, x_4\} \) of type \( D \) a pair of points \((x_1, x_2)\) and \((x_3, x_4)\) in the usual lattice \( \mathbb{N}^2 \).
Definition

For a fixed integer $k_0$, let us define $\mathcal{D}_n(k_0)$ the set of compositions \( \{x, y, z, y\} \) of type $\mathcal{D}$ of the integer $n$ such that $z - x = k_0$. A weak lattice path $\mathcal{L} \in \mathcal{D}$ from \( \{x, 0, y, 0\} \) to \( \{0, x, k_0, x\} \) containing all points in $\mathcal{D}(k_0)$ is defined in such a way that two adjacent vertices have the form:

\[
\{x, y, z, y\}, \{x - 1, y + 1, z - 1, y + 1\}
\]

thus, for each vertex there are two directions to get the next vertex, that is $\mathcal{L}$, $(-1, 0), (0, 1)$ or $(0, 1), (-1, 0)$. Henceforth, we let $\mathcal{L}_n(k_0)$ denote the set of all weak lattice paths linking out all the points in $\mathcal{D}_n(k_0)$. 

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Some Applications of the Representation Theory of Equipped Posets.
Definition

For a fixed integer $k_0$, let us define $\mathcal{D}_n(k_0)$ the set of compositions $\{x, y, z, y\}$ of type $\mathcal{D}$ of the integer $n$ such that $z - x = k_0$. A weak lattice path $\mathcal{L} \in \mathcal{D}$ from $\{x, 0, y, 0\}$ to $\{0, x, k_0, x\}$ containing all points in $\mathcal{D}(k_0)$ is defined in such a way that two adjacent vertices have the form:

$$\{x, y, z, y\}, \{x - 1, y + 1, z - 1, y + 1\}$$

thus, for each vertex there are two directions to get the next vertex, that is $\mathcal{L}, (-1, 0), (0, 1)$ or $(0, 1), (-1, 0)$. Henceforth, we let $\mathcal{L}_n(k_0)$ denote the set of all weak lattice paths linking out all the points in $\mathcal{D}_n(k_0)$. 
Theorem

For $k_0 \geq 1$ fixed, the number of weak lattice paths from $\{x,0,x+k_0,0\}$ to $\{0,x,k_0,x\}$ containing all points in $D_n(k_0)$ equals $2^x$. 

\[
\text{For } k_0 \geq 1 \text{ fixed, the number of weak lattice paths from } \\
\{x,0,x+k_0,0\} \text{ to } \{0,x,k_0,x\} \text{ containing all points in } D_n(k_0) \text{ equals } 2^x.
\]
Proof. For each \( y, 0 \leq y \leq x \), the ways to connect two adjacent vertices \( \{x, y, z, y\}, \{x - 1, y + 1, z - 1, y + 1\} \) are
\[ \{\{x, y, z, y\}, \{x - 1, y, z - 1, y\}, \{x - 1, y + 1, z - 1, y + 1\}\} \] and
\[ \{\{x, y, z, y\}, \{x, y + 1, z, y + 1\}, \{x - 1, y + 1, z - 1, y + 1\}\} \]. And the sequences of points in this case consists of the points,
\[ \{\{x, 0, k_0 + x, 0\}, \{x - 1, 1, k_0 + x - 1, 1\}, \ldots, \{0, x, k_0, x\}\} \]. \( \square \)
If we identify each partition \( \{x, y, z, y\} \) with a segment \((z, z') \subset \mathbb{N}^2\) with \(z = (x, y)\) and \(z' = (z, y)\) then on \(\mathbb{N}^2\) it is possible to obtain two types of lattice paths:

Weak lattice paths defined as above, linking segments and

Strong lattice paths where points are linked in the classical way, i.e., via using directions \((1, 0)\) and \((0, 1)\) or \((-1, 0)\) and \((0, -1)\).

The product of an strong lattice path with any other path is again strong. Whereas the product of weak lattice paths is again a weak lattice path.
Delannoy numbers

(0,6)  (1,6)  (2,6)  (3,6)  (4,6)  (5,6)  (6,6)
(0,5)  (1,5)  (2,5)  (3,5)  (4,5)  (5,5)  (6,5)
(0,4)  (1,4)  (2,4)  (3,4)  (4,4)  (5,4)  (6,4)
(0,3)  (1,3)  (2,3)  (3,3)  (4,3)  (5,3)  (6,3)
(0,2)  (1,2)  (2,2)  (3,2)  (4,2)  (5,2)  (6,2)
(0,1)  (1,1)  (2,1)  (3,1)  (4,1)  (5,1)  (6,1)
Special Weak Products

Special weak products happen when a weak lattice path $P$ from a segment $(z_0, z'_0)$ to a segment $(w_k, w'_k)$ is weakly multiplied with strong lattice paths starting in a point $(x_0, y_0)$ in such a case the product is given by a pair of classical lattice paths $(Q, Q')$ from $(x_0, y_0)$ to $z_0$ and $z'_0$ respectively (of course, the selection of the starting and ending points depends on the orientation).
Definition

Points and relations in \((\mathcal{D}, \sqsubseteq)\) are either weak or strong. We say that a point \(x \in \mathcal{D}\) is weak if and only if there exists a special weak product with lattice paths starting in \((0,0)\). Moreover, a chain \(C \subset \mathcal{D}\) is weak if all of its points are weak. Further, relations between points in \(\mathcal{D}\) with an strong point are also strong.
If we consider the natural representation

\[ U = (U_0; U_x \mid x \in \mathcal{D}_n(k_0)) \]

of the weak chain \( \mathcal{D}_n(k_0) \) on the underlying path algebra \( \Lambda \) defined by lattice paths connected \((0,0)\) with the remaining points \((x,y)\) in \( \mathbb{N}^2 \) and \( U_x^- = 0 \) for any \( p \in \mathcal{D}_n(k_0) \). Then \( \dim_{\Lambda^2} U_p \) equals the number \( d_p \) of special weak products from \((0,0)\) to the weak lattice path starting in \( p_0 = \{x_0, 0, k + k_0, 0\} \) and finishing in \( p \) for some fixed \( x_0 \).
Some Applications of the Representation Theory of Equipped Posets.
Some Applications of the Representation Theory of Equipped Posets.
Some Applications of the Representation Theory of Equipped Posets.
Theorem

For $k_0 \geq 1$ the dimension vector of the weak chain $D_n(k_0)$ is the vector $\dim_{\Lambda^2} = (d_0; d_p \mid p \in D_n(k_0))$ where for $p = \{x, y, x + k_0, y\} \in D_n(k_0)$;

$$d_p = 2^x c(x + y, y)c(x + k_0 + y, y).$$

$c(x, y) = \binom{x+y}{x}.$
**Proof.** For each \( p = \{x, y, x + k_0, y\} \in D_n(k_0)\), \( d_p \) is the number of special weak products from \( \{x_0, 0, x_0 + k_0, 0\} \) to \((0, 0)\) whose weak chain has as ending vertex \( p \). Since the number of weak lattice paths from \( \{x_0, 0, x_0 + k_0, 0\} \) to \( p \) is \( 2^x \) and the number of lattice paths from \((0, 0)\) to the segment \((x, y), (x + k_0, y))\) is \( c(x + y, x)c(x + k_0 + y, y) \), we are done. \( \square \)
Corollary

For $x_0$, $k_0 \geq 1$ fixed,

$$D(x_0, k_0) = \sum_{p \in \mathcal{D}_n(k_0)} d_p = \sum_{y=0}^{x_0} 2^y c(x_0 - y, y)c(x_0 + k_0 + y, y).$$
The last Theorem and Collorary allow to obtain a categorification of numbers $D(x_0, k_0)$ which can be seen as Dellannoy numbers $d(x, y)$. As we described before such numbers are also obtained by counting the number of lattice paths in $\mathbb{N}^2$ from $(0, 0)$ to $(x, y)$ considering directions $(1, 0), (0, 1)$ and $(1, 1)$. That is:

$$d(x, y) = d(x - 1, y) + d(x, y - 1) + d(x - 1, y - 1)$$

$$d(0, 0) = 1.$$
The last Theorem and Collorary allow to obtain a categorification of numbers $D(x_0, k_0)$ which can be seen as Delannoy numbers $d(x, y)$. As we described before such numbers are also obtained by counting the number of lattice paths in $\mathbb{N}^2$ from $(0, 0)$ to $(x, y)$ considering directions $(1, 0), (0, 1)$ and $(1, 1)$. That is:

$$d(x, y) = d(x - 1, y) + d(x, y - 1) + d(x - 1, y - 1)$$

$$d(0, 0) = 1.$$
Topological Data Analysis (TDA)
Definition (TDA)

*Topological Data Analysis (TDA)* is an approach to the analysis of datasets (for instance, Big Data) using techniques from topology. The idea is to analyze topological features across a family of spaces or point cloud data sets.
The simplex of dimension 0, 1, 2 and 3.

A simplicial complex of dimension 2 and 3.
A typical workflow runs as follows:

The input is a point cloud, that is, a finite subset of some Euclidean space or more generally a finite metric space.

After an initial filtering step (to remove undesirable points or to focus on high-density regions of the data).

A set of vertices is selected from the data and a simplicial complex $S$ is built on that vertex set, according to some prearranged rule.
A typical workflow runs as follows:

The input is a point cloud, that is, a finite subset of some Euclidean space or more generally a finite metric space.

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Data

Math World

Data

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Data

Math World

\[ f \]

\[ p \]
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Data

Math World

Data

$f^{-1}(p)$

$f$

$p$

$ullet$

Some Applications of the Representation Theory of Equipped Posets.
Data
Data

Math World

\[ f^{-1}(p) \]

\[ Data \]

\[ f \]

\[ q \]
Data

Math World

Data

\( f^{-1}(p) \)

\( f \)

\( q \)

\( p \)
In practice, the simplicial complex depends on a coarseness parameter $\varepsilon$ and it is obtained a nested family $\{S_\varepsilon\}_{\varepsilon \in [0, \infty]}$ which typically range from a discrete set of vertices at $S_0$ to a complex simplex at $S_\infty$. 
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G.E. Andrews's problems and Equipped posets

Auslander-Reiten quiver from a TDA

Digital Watermarking

References

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References

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The form of the data does matter
The form of the data does matter
**Persistent homology** takes the entire nested family \( \{S_\varepsilon\} \) and produces a barcode or persistence diagram as output.

**Definition**

A *barcode* is a collection of half-open subintervals \( [b_i, d_j) \subset [0, \infty) \) which describes the homology of the family as it varies over \( \varepsilon \).

An interval \( [b_j, d_j) \) represents a homological feature which is born at time \( b_j \) and dies at time \( d_j \).
If we discretise the variable $\varepsilon$ to a finite set of values. The family of simplicial complexes can be thought as a diagram of spaces:

$$S_1 \rightarrow S_2 \rightarrow \cdots \rightarrow S_n$$

where the arrows represent the inclusion maps.

If we apply the $k$-dimensional homology functor $H_k(-; k)$ with coefficients in a field $k$, this becomes a representation of a linear oriented quiver which is called a **persistence module**. Each possible barcode corresponds to an isomorphism type.
Silva et al use the Auslander-Reiten of Dynkin diagrams of type $A_n$ in order to give a generalization of persistence homology.
Categories of type \( \text{add } Y_t \)
In accordance with Ringel, given a Krull-Schmidt category $\mathcal{C}$, a full subcategory $\mathcal{D}$ of $\mathcal{C}$ closed under direct sums and summands (and isomorphisms) will be called and \textit{object class} in $\mathcal{C}$. We note that, an object class $\mathcal{D}$ is itself a Krull-Schmidt category and is uniquely determined by the indecomposable objects belonging to $\mathcal{D}$. Given a set of objects $N = \{N_1, N_2, \ldots, N_t\}$ in $\mathcal{C}$ then we let $\langle N \rangle$ denote the smallest object class containing $N$, it is given by the direct sums of direct summands of objects in $N$. 
We consider that object classes $M$ are Krull-Schmidt categories with short exact sequences belonging to $M$, in particular its Auslander-Reiten quiver is defined in such a way that if an indecomposable object $U \in \Gamma(M)$ then its translate $\tau_M(U)$ belong to $\Gamma(M)$ as well. The Auslander-Reiten quiver $\Gamma(M)$ is called a relative Auslander-Reiten quiver.
Categories of type \( \text{add } Y_t \)

Equipped Posets of type \( Y_t \)
We describe the Auslander-Reiten quiver of subcategories of representations of some equipped posets.

Definition

If \( P \) is an equipped poset with \( 3t - 2 \) points such that
\[
P = \{1 \prec 2 \prec \cdots \prec t - 1\} + \{t \prec t + 1 \prec t + 2 \prec \cdots \prec 2t - 2 \prec 2t - 1\} + \{2t \prec 2t + 1 \prec 2t + 2 \prec \cdots \prec 3t - 3 \prec 3t - 2\}
\]
with \( t - 1 \prec t, t \prec 2t \) then \( P \) is said to be an equipped posets of type \( Y_t \).
Henceforth, we let $\text{add } Y_t$ denote the object class in a category of representations $\text{rep } \mathcal{P}$ of an equipped poset $\mathcal{P}$ of type $Y_t$ such that:

$$
\text{add } Y_t = \langle P(k), T(i), T(i,j), P(i,l), G_h(i,l) \rangle, \\
1 \leq i \leq 2t - 1, \\
1 < j \leq 2t - 1, \\
h \in \{1, 2\} \\
1 \leq k \leq 3t - 2, \\
2t \leq l \leq 3t - 2 \quad \text{if} \quad t + 1 \leq i \leq 2t - 1,
$$
(2)
Categories of type $\text{add } Y_t$

The following is an example of the diagram of a equipped Posets of type $Y_t$.

$\mathcal{P} =$
Categories of type \( \text{add } Y_t \)

**Theorem**

The following statements hold for an equipped Posets of type \( Y_t = \mathcal{P} \):

1. If \( t = 2 \) then \( \mathcal{P} \) is finite representation type;
2. If \( t = 3 \) then \( \mathcal{P} \) is finite growth representation type;
3. If \( t \geq 4 \) then \( \mathcal{P} \) is wild representation type;
Proof. The proof is center in the evolvent of $\mathcal{P}$ as follows:

If $t = 2$ then the diagrams of $\mathcal{P}$ and its corresponding evolvent $\hat{\mathcal{P}}$ have the following forms:

$\mathcal{P} = \begin{array}{c}
\otimes 3 \\
\otimes 2 \\
\otimes 1 \\
\end{array}$

$\hat{\mathcal{P}} = \begin{array}{c}
\circ 3' \\
\circ 2' \\
\circ 1' \\
\end{array}$

$\circ 4$

$\circ 2''$

$\circ 1''$

$\mathcal{P}$ is of finite representation type provided that $\hat{\mathcal{P}}$ does not contain Kleiner’s critical.
If \( t = 3 \) then the diagram of \( \mathcal{P} \) and its corresponding evolvent \( \hat{\mathcal{P}} \) have the following forms:

\[
\begin{align*}
\mathcal{P} = & \quad 5 \quad \bigcirc \quad 4 \quad 
\bigcirc \quad 3 \quad \bigcirc \quad 2 \quad \bigcirc \quad 1 \\
\hat{\mathcal{P}} = & \quad 1'' \quad \bigcirc \quad 2'' \quad \bigcirc \quad 3'' \quad \bigcirc \quad 6 \quad \bigcirc \quad 4'' \quad \bigcirc \quad 5'' \\
\end{align*}
\]

In this case, \( \hat{\mathcal{P}} \) contains as subposet \( \{4' < 5', \ 6 < 7, \ 4'' < 5''\} \) thus \( \mathcal{P} \) is of infinite representation type. Moreover, since \( \hat{\mathcal{P}} \) contains no Nazarova’s critical \( N_1, \ldots, N_6 \) then \( \mathcal{P} \) is of tame representation type thus \( \mathcal{P} \) is of finite growth representation type provided that \( \mathcal{P} \) contains no critical of type \( G_1, \ldots, G_7 \).
If $t \geq 4$ then diagrams of $\mathcal{P}$ and corresponding evolvent $\hat{\mathcal{P}}$ have the following forms:
As a consequence of L. A. Nazarova’s Theorem we conclude that $\mathcal{P}$ is of wild representation type provided that $\widehat{\mathcal{P}}$ contains the subposet $\{(t + 1)' < (t + 2)', 2t < 2t + 1, (t + 1)'' < (t + 2)'' < (t + 3)''\} = (2, 2, 3) = N_3$. □
The following theorem characterizes monomorphisms and epimorphisms in subcategories $\text{add } Y_t$. 
Categories of type $\text{add } Y_t$

**Theorem**

Let $\varphi : U \to V$ be a not null morphism in $\text{add } Y$ with $U, V \in Y$, then

1. $\varphi$ is a monomorphism if and only if it satisfies one of the following conditions:
   
   **1**
   
   $U = P(i), V = P(j); \quad i, j \in 1^\vee$ or $i, j \in 1^\wedge$.  
   $U = P(i), V = T(j, k); \quad i \in k^\vee$.  
   $U = P(i), V = P(j, k); \quad i \in k^\vee + j^\wedge$.  
   $U = P(i), V = G_h(k, j); \quad i \geq j \in t^\wedge$.  

   **II**
   
   $U = T(i), V = T(j); \quad i \geq j$.  
   $U = T(i), V = T(k, j); \quad i \geq k$.  
   $U = T(i), V = G_h(k, j); \quad i \geq k$.  

   **III**
   
   $U = T(i, j), V = T(k, l); \quad i \in k^\vee, j \in l^\vee$.  

   **IV**
   
   $U = P(i, j), V = P(k, l); \quad i \in k^\vee, j \in l^\wedge$.  
   $U = P(i, j), V = P(k); \quad k \in t_\perp$.  

   **V**
   
   $U = G_h(i, j), V = G'_h(k, l); \quad h \leq h', i > k, j \leq l$.  

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Some Applications of the Representation Theory of Equipped Posets.
### Theorem

φ is an epimorphism if and only if it satisfies one of the following conditions:

- **a** \( U = V = P(i); \)
- **b** \( U = T(i), V \in \{P(i), T(i)\}; \)
- **c** \( U = T(i, j), V \in \{P(i), T(i, j)\}; \)
- **d** \( U = V = P(i, j); \)
- **e** \( U = G_h(i, j), V \in \{P(i, j), G_h(i, j)\}. \)
Corollary

*For an equipped poset $\mathcal{P}$ of type $Y_t$ the following results hold:*

1. $P(j), j \in t^\wedge$, and $T(i)$ for $i \in (2t - 1)^\vee$ are the only indecomposable projective representations of $\text{add } Y_t$.

2. $P(1)$ is the only indecomposable injective representation of $\text{add } Y_t$. 
Corollary

For a morphism $\varphi : U \rightarrow V \in \text{add } Y_t(U, V)$ it holds that:

1. $\varphi : P(i) \rightarrow P(j)$ is irreducible if and only if $i = j + 1$, $i, j \in t^\updownarrow$,

2. $\varphi : T(i, j) \rightarrow P(i)$, $\varphi : T(i + 1, j) \rightarrow T(i, j)$, $\varphi : T(i + 1, j) \rightarrow T(i, j - 1)$ are irreducible morphisms,

3. $\varphi : T(i) \rightarrow T(i, 2t - 1)$, $\varphi : T(i) \rightarrow T(i, i - 1)$, are irreducible morphisms,

4. $\varphi : P(k, l) \rightarrow T(k - 1, l - 1)$, $k \in t^\uparrow$, $l \in t^\downarrow$, $P(k, l) \rightarrow P(k - 1, l)$, $P(k, l) \rightarrow P(k, l - 1)$, are irreducible morphisms,
Corollary

5 \( \varphi : G_1(k, l) \to G_1(k, l - 1), \varphi : G_1(k, l) \to G_1(k - 1, l), \varphi : G_1(k, l) \to G_2(k, l) \) are irreducible morphisms,

6 \( \varphi : G_2(k, l) \to G_2(k, l - 1), \varphi : G_2(k, l) \to G_2(k - 1, l), \varphi : G_2(k, l) \to P(k, l), \varphi : P(i) \to G_1(2t - 1, i) \) for \( i \in t^\nabla \) are irreducible morphisms.

7 \( \varphi : P(2t - 1) \to P(2t - 1, 3t - 2), \) is an irreducible morphism.
Categories of type $\text{add } Y_t$
Auslander-Reiten quiver from a TDA point of view
Let $d_U$ denote the dimension vector of the indecomposable representation $U$, i.e., $d_U = \text{dim}(U) = (d_0, d_x \mid x \in \mathcal{P})$. 

$$
P(2) = 
\begin{array}{c}
G \otimes 5 \\
G \otimes 4 \\
G \otimes 3 \\
G \otimes 2 \\
0 \otimes 1
\end{array}
$$

$$
d_{P(2)} = (1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1)
$$
Let $d_U$ denote the dimension vector of the indecomposable representation $U$, i.e., $d_U = \text{dim}(U) = (d_0, d_x \mid x \in \mathcal{P})$.

\[
\begin{array}{c}
P(2) = \\
G \quad 5 \\
G \quad 4 \\
G \quad 3 \\
G \quad 2 \\
0 \quad 1 \\
\end{array}
\]

\[
\begin{array}{c}
G \quad 7 \\
G \quad 6 \\
G \\
\end{array}
\]

$d_{P(2)} = (1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1)$
Categories of type add $Y_t$

$$T(1) = \langle (1, u)^t \rangle_1 \xrightarrow{2} \langle (1, u)^t \rangle_2 \xrightarrow{3} \langle (1, u)^t \rangle_3 \xrightarrow{4} \langle (1, u)^t \rangle_4 \xrightarrow{5} \langle (1, u)^t \rangle_5 \xrightarrow{6} \langle (1, u)^t \rangle_6 \xrightarrow{7} G^2$$

$$d_{T(1)} = (2 \ 1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 2)$$
Categories of type $\text{add } Y_t$

$$\langle (1, u)^t \rangle \otimes 5$$

$$\langle (1, u)^t \rangle \otimes 4$$

$$\langle (1, u)^t \rangle \otimes 3$$

$$\langle (1, u)^t \rangle \otimes 2$$

$$\langle (1, u)^t \rangle \otimes 1$$

$$T(1) =$$

$$d_{T(1)} = (2 \ 1 \ 1 \ 1 \ 1 \ 1 \ 2 \ 2)$$
We define the weight of $d_U$ as

$$w(d_U) = \sum_{x \in \mathcal{P} \cup \{0\}} d^U_x$$

in such a way that the distance $d(U, V)$ between two representations $U, V \in \text{add } Y$ is given by the formula

$$d(U, V) = \sum_{x \in \mathcal{P} \cup \{0\}} |d^U_x - d^V_x|$$
Theorem

Let $U, V \in \text{add } Y$ be, if $d(U, V) = w(d_X^{U-V}) = 1$ then there is an irreducible morphism from $V$ to $U$.
With this distance we can define the next ball.

\[ B(U, 1) = \{ V : d(U, V) \leq 1 \} = \{ V \neq P(k), \ k \in 1^\gamma : d(U, V) \leq 1 \} \]

**Theorem**

If \( U \in B(V, 1) \) and \( d(U, V) = w(d_x^{-V}) \) then there exists an irreducible morphism from \( V \) to \( U \).

If \( A = \{ P(k, j) \} \) and \( B = \{ P(1, j) \} \). Now we defined \( d(A, B) = \min \{ d(U, V), U \in A, V \in B \} \).

**Theorem**

If \( d(A, B) = d(U_0, V_0) \) then there exists an irreducible morphism from \( V_0 \) to \( U_0 \).
With this distance we can define the next ball.

\[ B(U, 1) = \{ V : d(U, V) \leq 1 \} = \{ V \neq P(k), \ k \in 1^\gamma : d(U, V) \leq 1 \} \]

**Theorem**

*If \( U \in B(V, 1) \) and \( d(U, V) = w(d_x^{U-V}) \) then there exists an irreducible morphism from \( V \) to \( U \).*

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**Theorem**

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**Theorem**

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If \( A = \{ P(k, j) \} \) and \( B = \{ P(1, j) \} \). Now we defined \( d(A, B) = \min\{d(U, V), U \in A, V \in B\} \).

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Categories of type $\text{add } Y_t$
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Digital watermarks
Digital watermarks complement the encryption work to enable security in the transmission of data through the internet (images, texts). A digital watermark is a piece of information hiding into the content of a message in such a way that it will be imperceptible to the human eye but easy to detect by a computer. The advantage here is that the digital watermark cannot be separated from the content.
The theory of algorithms of differentiation is a theoretical framework for digital watermarking procedures. In this case the algorithm DVII is applied to representation of equipped posets defined over an image algebra.
Now, we use representations of equipped poset \( \mathcal{P} = \{a \prec c_1 \prec c_2 \prec c_3\} + \{b\} \) and its corresponding derived poset in order to embed digital watermarks in a system of images. To do that, we define the following representation over the pair \((\mathbb{R}, \mathbb{C})\), we assume the notation \(k(e_i, e_j) = ke_i + ki e_j\), where \(e_i\) and \(e_j\) are images and \(k \in \mathbb{R}\) is chosen in such a way that a user only can see only one of the two images.
Original image, digital watermark and watermarked image.
In image algebra a system of images \( k(e_i, e_j) \) is interpreted as a sum of the form \( ke_i + ke_j \), for any number \( k \in \mathbb{R} \).
Consider the representation of $\mathcal{P}$ over the image algebra given by the following formulae:

\[
U_a = \sum_{i_1} \alpha_{i_1} (e_{i_1}, e_{j_1}) + \sum_{s_1} \beta_{s_1} h_{s_1},
\]

\[
U_{c_1} = U_a + \sum_{i_2} \alpha_{i_2} (e_{i_2}, e_{j_2}) + \sum_{s_2} \beta_{s_2} h_{s_2},
\]

\[
U_{c_2} = U_{c_1} + \sum_{i_3} \alpha_{i_3} (e_{i_3}, e_{j_3}) + \sum_{s_3} \beta_{s_3} h_{s_3},
\]  \hspace{1cm} (3)

\[
U_{c_3} = U_{c_2} + \sum_{i_4} \alpha_{i_4} (e_{i_4}, e_{j_4}) + \sum_{s_4} \beta_{s_4} h_{s_4}
\]

\[
U_b = \sum_{i_1} \gamma_{i_1} e_{i_1} + \sum_{t=1}^{4} \sum_{s_t} \beta_{s_t} h_{s_t}.
\]
Consider the representation of \( \mathcal{P} \) over the image algebra given by the following formulae:

\[
U_a = \sum_{i_1} \alpha_{i_1}(e_{i_1}, e_{j_1}) + \sum_{s_1} \beta_{s_1} h_{s_1},
\]

\[
U_{c_1} = U_a + \sum_{i_2} \alpha_{i_2}(e_{i_2}, e_{j_2}) + \sum_{s_2} \beta_{s_2} h_{s_2},
\]

\[
U_{c_2} = U_{c_1} + \sum_{i_3} \alpha_{i_3}(e_{i_3}, e_{j_3}) + \sum_{s_3} \beta_{s_3} h_{s_3},
\]

\[
U_{c_3} = U_{c_2} + \sum_{i_4} \alpha_{i_4}(e_{i_4}, e_{j_4}) + \sum_{s_4} \beta_{s_4} h_{s_4},
\]

\[
U_b = \sum_{i_1} \gamma_{i_1} e_{i_1} + \sum_{t=1}^{4} \sum_{s_t} \beta_{s_t} h_{s_t}.
\]

\[
\sum_{i_1} \gamma_{i_1} e_{i_1} = \sum_{i_1} \alpha_{i_1} e_{i_1} + \sum_{i_1} \alpha_{i_1} e_{i_1} + \sum_{i_1} \alpha_{i_1} e_{i_1} + \sum_{i_1} \alpha_{i_1} e_{i_1} + \sum_{i_1} \alpha_{i_1} e_{i_1}.
\]
In this case, subspace $U_b$ contains digital watermarks of images $e_{jr}$, $0 \leq r \leq 3$ ($c_0 = a$). Algorithm of differentiation VII allows to extracted such watermarks, to do that each subspace $U'_{c_i^+}$ is defined as follows:

\[
U'_{a^+} = \sum_{r=0}^{3} \sum_{i_r^{c_r}} \alpha_{i_r^{c_r}} e_{i_r^{c_r}}
\]

\[
U'_{c_r^+} = U'_{a^+} + \sum_{i_r} \alpha_{i_r} (e_{i_r}, e_{jr}) + \sum_{s_r} \beta_{s_r} h_{s_r}, \quad 1 \leq r \leq 3, \quad (5)
\]

\[
U'_{c_r^-} = \sum_{s_r} \beta_{s_r} h_{s_r}, \quad 0 \leq r \leq 3.
\]
In this case, subspace $U_b$ contains digital watermarks of images $e_{jr}$, $0 \leq r \leq 3$ ($c_0 = a$). Algorithm of differentiation VII allows to extracted such watermarks, to do that each subspace $U'_{c_i^+}$ is defined as follows:

$$U'_{a^+} = \sum_{r=0}^{3} \sum_{i_1^{cr}} \alpha_{i_1^{cr}} e_{i_1^{cr}}^c$$

$$U'_{c_r^+} = U'_{a^+} + \sum_{i_r} \alpha_{i_r} (e_{i_r}, e_{jr}) + \sum_{s_r} \beta_{s_r} h_{s_r}, \quad 1 \leq r \leq 3, \quad (5)$$

$$U'_{c_r^-} = \sum_{s_r} \beta_{s_r} h_{s_r}, \quad 0 \leq r \leq 3.$$
Coefficients in \( U^+_a \) are chosen in such a way that the only image that an user can see is one of the type \( e_{i_1^c r} \), \( 0 \leq r \leq 3 \). On the other hand, digital watermarks of the form \( \sum_{s_t} \beta_{s_t} h_{s_t} \) are given by subspaces \( U'_{c r} \).

And images \( \sum_{i_1^c r} \alpha_{i_1^c r} e_{i_1^c r} \), \( 0 \leq r \leq 3 \) are the recovered digital watermarks from subspace \( U_{c r} \).
Since this procedure is independent of domain, it can be applied in spatial and frequency domain. In MATLAB routines was implemented to embed system of digital watermarks.

A MATLAB routine interprets DVII as a system of digital watermarks in frequency domain. To do that, a singular value decomposition is applied after a wavelet procedure in order to embed a digital watermark.
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References


Thank You