# Optimal Control Policy for Batch Demand Supply Chains

Jin Wang
Department of Mathematics and Computer Science
Valdosta State University
Valdosta, GA 31698-0040

Abhijit Deshmukh
Department of Mechanical and Industrial Engineering
University of Massachusetts
Amherst, MA 01003–2210

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**Abstract**

We study a single product batch demand serial supply chain with Poisson arrivals and exponential manufacturing times. Each order has a fixed due date and demand quantity. A general model of two facilities in tandem with feedbacks is discussed. A product has a probability of acceptance at each stage. A rejected product can cycle through the facility a finite number of times. We derive the expected number of products in warehouses of facilities 1 and 2 respectively and the total expected work-in-process inventory. A total cost model is introduced as a nonlinear programming problem. The objective is to minimize the total cost with respect to the production rates and the total manufacturing time. We derive the optimal manufacturing rates, the raw material arrival rate, and the optimal total inventory and production costs. We analyze the manufacturing rates and objective function to changes in due dates and batch demands. Finally, we present computational results based on real world data.

KEY WORDS: Queueing; Stochastic Process; Supply Chain; Optimal Control; Optimization; Sensitivity Analysis.

# 1 Introduction

In this paper, we model a batch demand serial supply chain, as shown in Figure 2.1, which is motivated by the practice at a major automobile manufacturer. The final demand from the automobile assembly plants is given to the sub–system manufacturers as batched forecasts for each automobile configuration. We consider the problem of determining production rates at the sub–system manufacturing facilities given random yields, rework and unpredictable raw material supply. This problem is different from the production control problem. The detailed discussion of controlling production can be founded in Glassey, Seshadri, and shanthikumar [1996, 1998]. We specifically consider the case of two sub–system manufacturing facilities with intermediate storage locations and final inventory storage. The yield process of each manufacturing facility is Poisson. A fixed quantity, D, has to be delivered at a given time T. This is the only order to be delivered at time T. To facilitate analytical treatment, we assume that the arrival process to the first manufacturing facility is Poisson.The Monte Carlo simulation results show that our method is a good approximation approach when this Poisson assumption is relaxed. The focus of this research is on determining optimal manufacturing rates and raw material order rate in order to minimize the total inventory and production costs. We also discuss the sensitivity of the optimal control policy with respect to the demand variations and due date changes.

Considerable advances have been made in analyzing and modeling serial supply chain problem under different conditions and objectives. Veatch and Wein [1994] study a two-station tandem production/inventory system without feedback. They focus on controlling the production rates in order to minimize the inventory holding and backordering costs. Duenyas and Patana-Anake [1998] investigate the performance of a simple base stock policy for a multiple-stage tandem production/inventory system producing a single product. Jeong and Kim [1998] discuss an method to evaluate performance of tree-structured assembly/disassembly systems with finite buffer capacity. Conway, Maxwell, McClain, and Thomas [1988] study the role of work-in-process inventory in serial production lines. Yano and Lee [1995] present an excellent survey on the study of lot sizing with random yield. Their survey does not discuss the work related to quality control procedures and performance evaluation models based on queueing networks. Sobel [1994] investigates a multistage manufacturing problem in which quality influences lot sizes. The decision problem is to decide how many units to process at each stage of manufacturing. For the same fundamental decision problem as in Sobel [1994], Sinha and Sobel [1994] specify an optimal decision rule that overcomes the computational complexity. Graves [1987] provides a survey on determining production and inventory policies when demand is uncertain. It provides an excellent review of production planning models for multi-location networks. Garg [1996] describes an application of designing products and processes for supply chain management at a large electronics products manufacturer. The objective is reduce the costs of complexity resulting from a proliferation of parts and processes in the manufacturer's supply chain.

Most of the research so far has focused on the situations when the number of times a product can be reworked is unlimited or the manufacturing quality is perfect without any feedback. We study a general model of two facilities in tandem with feedbacks. The number of times for rework is limited which is a more reasonable assumption for general manufacturing systems. This paper is organized as follows. In Section 2, we analyze a general model of two facilities in tandem with feedbacks, for batched demand. In Section 3, we derive the optimal production rate for minimum work–in–process for two different objectives. In Section 4, we derive inventory cost of demand orders and the raw material supply rate. A total cost model is introduced as a nonlinear programming problem. In Section 5, we study the performance bounds for two different situations, where the demand varies by a fixed percentage or we are constrained by confidence level for satisfying demand. Section 6 presents the sensitivity analysis with respect to due date and demand changes. In Section 7, we present computational results for the supply chain based on masked data from a major automobile manufacturer. Our computational results are consistent with the theoretical results from the sensitivity analysis in Section 6. Finally, Section 8 presents the summary of main results in this paper and directions for future research.

## 2 Serial Supply Chain Model

We consider two manufacturing facilities in tandem with feedbacks, as shown in Figure 2.1. The raw material arrives at the first facility as a Poisson process with rate $\lambda$ (Karlin and Taylor [1975]). The raw material is stored in a warehouse before it is processed at the first facility. Upon completion of processing at the first stage, the product may be rejected and sent back to the first facility with probability $p_1$. With probability $1 - p_1$, the product waits for processing at the second facility. Similar to the first stage, products may be sent back to the second facility after processing with probability $p_2$ and the product is accepted as finished product with probability $1-p_2$, The finished goods are stored in a finished product inventory before the batch is shipped to the customers. At facility $i$, a product is only allowed to be processed $n_i$ times, for $i = 1, 2$. The production rate of facility $i$ is exponentially distributed with rate $\mu_i$, for $i = 1, 2$. We assume the raw material supply rate, production rates at the two facilities, and the yield rates are all independent. We also assume that the queueing network has a steady state distribution. Most of the basic queueing results can be found in Ross [1993], Kleinrock [1975] and Little [1961]. More advanced topics can be found in Walrand [1988], Buzacott and Shanthikumar [1993], Pinedo [1995] and Wolff [1989].

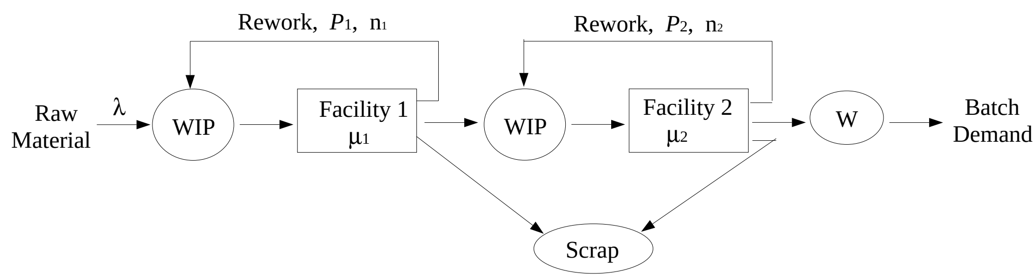


Figure 2.1: Two manufacturing facilities in tandem with feedbacks.

In this model, the customer demand $D$ is a fixed quantity, occurs at a fixed time point, and has to be delivered at a given time $t_0$. The total arrival rate into facility 1 is

$$\lambda_1 = \lambda + \lambda p_1 + \lambda p_1^2 + \cdots + \lambda p_1^{n_1-1} = \frac{\lambda(1-p_1^{n_1})}{1-p_1}.$$

Here we have $n_1$ different arrival processes including one outside supplier process and $n_1 - 1$ rework processes. The $i$th arrival process has rate $\lambda p_1^{i-1}$ for $i = 1, \ldots, n_1$. The utilization of facility 1 is

$$\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{\lambda}{\mu_1}\left(\frac{1-p_1^{n_1}}{1-p_1}\right).$$

The expected numbers of products waiting in queue before facility 1 is

$$W_1 = \frac{\rho_1}{1-\rho_1} = \frac{\lambda(1-p_1^{n_1})}{\mu_1(1-p_1) - \lambda(1-p_1^{n_1})}.$$

For each arrival process at facility 1, the probability of acceptance as a good product after processing is $1 - p_1$. Thus, the output of facility 1 is a Poisson process with rate

$$\lambda_1(1-p_1) = \lambda(1-p_1^{n_1}),$$

which is the rate of arrival from facility 1 to facility 2. Similarly to the discussion of $\lambda_1$, we have the total arrival rate at facility 2

$$\lambda_2 = \frac{\lambda(1-p_1^{n_1})(1-p_2^{n_2})}{1-p_2}.$$

The utilization of facility 2 is

$$\rho_2 = \frac{\lambda_2}{\mu_2} = \frac{\lambda}{\mu_2}\frac{(1-p_1^{n_1})(1-p_2^{n_2})}{1-p_2}.$$

The expected numbers of products waiting in queue before facility 2 is

$$W_2 = \frac{\rho_2}{1-\rho_2} = \frac{\lambda(1-p_1^{n_1})(1-p_2^{n_2})}{\mu_2(1-p_2)-\lambda(1-p_1^{n_1})(1-p_2^{n_2})}.$$

The output of facility 2 is a Poisson process with rate

$$\lambda_2(1-p_2) = \lambda(1-p_1^{n_1})(1-p_2^{n_2}).$$

The total expected work-in-process inventory is

$$W = W_1 + W_2 = \frac{\lambda(1-p_1^{n_1})}{\mu_1(1-p_1)-\lambda(1-p_1^{n_1})} + \frac{\lambda(1-p_1^{n_1})(1-p_2^{n_2})}{\mu_2(1-p_2)-\lambda(1-p_1^{n_1})(1-p_2^{n_2})}. \qquad (2.1)$$

# 3 Optimal Production Rates for Minimum Work–in–Process

We consider the general model which was introduced in the Section 2. We assume that the production rates at each facility can be controlled and the raw material supply rate is fixed. Based on the analysis in Section 2, we derive the optimal production rates for two different objectives, minimum work–in–process and minumum cost.

## 3.1 Minimizing Work–in–Process

We consider the objective of minimizing the total expected work-in-process inventory with respect to the production rates $\mu_1$ and $\mu_2$. From (2.1), the total expected work-in-process inventory is

$$W(\mu_1,\mu_2) = \frac{\lambda(1-p_1^{n_1})}{\mu_1(1-p_1)-\lambda(1-p_1^{n_1})} + \frac{\lambda(1-p_1^{n_1})(1-p_2^{n_2})}{\mu_2(1-p_2)-\lambda(1-p_1^{n_1})(1-p_2^{n_2})}.$$

The system has a stationary distribution if and only if $\rho_1 < 1$ and $\rho_2 < 1$. It is equivalent to the following two conditions

$$\mu_1 > \frac{\lambda(1-p_1^{n_1})}{1-p_1} \quad \text{and} \quad \mu_2 > \frac{\lambda(1-p_1^{n_1})(1-p_2^{n_2})}{1-p_2}.$$

We also assume that the production rates $\mu_1$ and $\mu_2$ are limited and bounded by $\mu_1^0$ and $\mu_2^0$, respectively. The optimization problem is

$$
\begin{array}{lc}
\text{minimize} & W(\mu_1,\mu_2) = \frac{\lambda(1-p_1^{n_1})}{\mu_1(1-p_1)-\lambda(1-p_1^{n_1})} + \frac{\lambda(1-p_1^{n_1})(1-p_2^{n_2})}{\mu_2(1-p_2)-\lambda(1-p_1^{n_1})(1-p_2^{n_2})} \\
\text{subject to} & \mu_1 > \frac{\lambda(1-p_1^{n_1})}{1-p_1}; \\
& \mu_2 > \frac{\lambda(1-p_1^{n_1})(1-p_2^{n_2})}{1-p_2}; \\
& \mu_1 \le \mu_1^0; \\
& \mu_2 \le \mu_2^0; \\
& \mu_1,\ \mu_2 > 0.
\end{array}
\tag{3.1}
$$

Here $W(\mu_1,\mu_2)$ is a decreasing function in both $\mu_1$ and $\mu_2$. Therefore the optimal solution of (3.1) is

$$
\begin{cases}
\mu_1^* = \mu_1^0; \\
\mu_2^* = \mu_2^0.
\end{cases}
$$

## 3.2 Minimizing Work–in–Process and Capacity Expansion Costs

In this section, our objective is to minimize the total cost which includes the work-in-process and capacity expansion costs for both facilities. We assume that the unit inventory cost at facility $i$ is $c_{w_i}$ per unit time and the unit capacity expansion cost at facility $i$ is $c_{m_i}$, which is the cost of increasing the production rate. The total capacity expansion costs at facility 1 and facility 2 are

$$c_{w_1}W_1(\mu_1) + c_{w_2}W_2(\mu_2),$$

where $W_1$ and $W_2$ are obtained from Section 2. The total capacity expansion costs are

$$c_{m_1}\mu_1 + c_{m_2}\mu_2.$$

Therefore the optimization problem can be stated as

$$\begin{aligned}
\text{minimize} \qquad & C(\mu_1,\mu_2) = c_{w_1}\frac{\lambda(1-p_1^{n_1})}{\mu_1(1-p_1)-\lambda(1-p_1^{n_1})} + c_{w_2}\frac{\lambda(1-p_1^{n_1})(1-p_2^{n_2})}{\mu_2(1-p_2)-\lambda(1-p_1^{n_1})(1-p_2^{n_2})} \\
& +c_{m_1}\mu_1 + c_{m_2}\mu_2 \\
\text{subject to} \qquad & \mu_1 > \frac{\lambda(1-p_1^{n_1})}{1-p_1}; \\
& \mu_2 > \frac{\lambda(1-p_1^{n_1})(1-p_2^{n_2})}{1-p_2}; \\
& \mu_1,\ \mu_2 > 0.
\end{aligned} \tag{3.2}$$

The partial derivatives of $C(\mu_1,\mu_2)$ with respect to $\mu_1$ and $\mu_2$ are

$$\frac{\partial}{\partial\mu_1}C(\mu_1,\mu_2) = -\frac{c_{w_1}\lambda(1-p_1^{n_1})(1-p_1)}{[\mu_1(1-p_1)-\lambda(1-p_1^{n_1})]^2} + c_{m_1}$$

and

$$\frac{\partial}{\partial\mu_2}C(\mu_1,\mu_2) = -\frac{c_{w_2}\lambda(1-p_1^{n_1})(1-p_2^{n_2})(1-p_2)}{[\mu_2(1-p_2)-\lambda(1-p_1^{n_1})(1-p_2^{n_2})]^2} + c_{m_2}.$$

Let

$$\left\{\begin{array}{rcl} \frac{\partial}{\partial\mu_1}C(\mu_1,\mu_2) & = & 0, \\ \frac{\partial}{\partial\mu_2}C(\mu_1,\mu_2) & = & 0. \end{array}\right. \tag{3.3}$$

Solving (3.3), we obtain its solution

$$\left\{\begin{array}{rcl} \mu_1^* & = & \sqrt{\frac{c_{w_1}\lambda(1-p_1^{n_1})}{c_{m_1}(1-p_1)}} + \frac{\lambda(1-p_1^{n_1})}{1-p_1}, \\ \mu_2^* & = & \sqrt{\frac{c_{w_2}\lambda(1-p_1^{n_1})(1-p_2^{n_2})}{c_{m_2}(1-p_2)}} + \frac{\lambda(1-p_1^{n_1})(1-p_2^{n_2})}{1-p_2}. \end{array}\right. \tag{3.4}$$

All items of $C(\mu_1,\mu_2)$ are convex functions of $\mu_1$ and $\mu_2$. Here $C(\mu_1,\mu_2)$ is convex since it is a sum of convex functions. For details of the verification of convexity, similar problems are solved in Buzacott and Shanthikumar [1992]. Therefore the optimal production rates are

$$\left\{\begin{array}{rcl} \mu_1^* & = & \sqrt{\frac{c_{w_1}\lambda(1-p_1^{n_1})}{c_{m_1}(1-p_1)}} + \frac{\lambda(1-p_1^{n_1})}{1-p_1}, \\ \mu_2^* & = & \sqrt{\frac{c_{w_2}\lambda(1-p_1^{n_1})(1-p_2^{n_2})}{c_{m_2}(1-p_2)}} + \frac{\lambda(1-p_1^{n_1})(1-p_2^{n_2})}{1-p_2}. \end{array}\right.$$

# 4 Optimal Production Rates for Batch Demands

We now discuss a model where the customer demands have to be satisfied in batches, which is shown in Figure 2.1. The objective is to minimize the total cost with respect to decision variables $\mu_1$, $\mu_2$, and $t$. The total cost over the time window $[0, t]$ includes the inventory holding cost at facility 1 and facility 2, the capacity expansion costs at facility 1 and facility 2, and the inventory holding cost for finished goods before the batch demand orders are shipped.

## 4.1 Inventory Cost of Demand Orders

We assume that the batch demand order quantity is $D$, which is due at time $t_0$. From Section 2, the output from facility 2 is a Poisson process. In order to meet the due date $t_0$, the expected number of arrivals over the time window $[0,\ t_0]$ must be equal to the total demand $D$, i.e.,

$$\mathrm{E}[N(t_0)] = D.$$

Since $\{N(t), t \geq 0\}$ is a Poisson process, then

$$\lambda_d t_0 = D,$$

where $\lambda_d$ is the rate of the Poisson process. This rate must be

$$\lambda_d = \frac{D}{t_0}.$$

The total expected number of arrivals by time $t_0$ is

$$\mathrm{E}[N(t_0)] = \lambda_d t_0 = \frac{D}{t_0} t_0 = D.$$

All arrivals leave the system at the same time $t_0$, which is the due date. In general, the total expected number of arrivals by time $t$ ($t \in [0, t_0]$) is

$$\mathrm{E}[N(t)] = \lambda_d t = \frac{D}{t_0} t.$$

The total expected inventory holding time of $D$ products is

$$\int_0^{t_0} \mathrm{E}[N(t)] dt = \int_0^{t_0} \frac{D}{t_0}\ t\ dt = \frac{D\ t_0}{2}.$$

The expected inventory holding time for each arrival over the time window $[0, t_0]$ is

$$\frac{\int_0^{t_0} \mathrm{E}[N(t)]dt}{D} = \frac{t_0}{2}.$$

We assume that the unit inventory cost of demand orders per unit time is $c_{w_3}$. Therefore, the total expected inventory cost of batch demand orders is $\dfrac{c_{w_3} D\ t_0}{2}$.

## 4.2 Raw Material Supply Rates

In this section, we derive the optimal raw material supply rates for the models introduced in Section 2 with respect to the demand rate $\lambda_d$ over a time window $[0, t_0]$. From Section 4.1, if $D$ is the total demand with the due date $t_0$, then the demand rate is

$$\lambda_d = \frac{D}{t_0}.$$

We need to determine the raw material supply rate $\lambda$ which will result in the output process with rate $\lambda_d$. The output of facility 2 is a Poisson process with rate $\lambda(1 - p_1^{n_1})(1 - p_2^{n_2})$. Let

$$\lambda(1 - p_1^{n_1})(1 - p_2^{n_2}) = \lambda_d,$$

where $\lambda_d = D/t_0$. Then the raw material supply rate to facility 1 must be at least

$$\lambda = \frac{D}{(1 - p_1^{n_1})(1 - p_2^{n_2})\ t_0}. \tag{4.1}$$

## 4.3 Total Cost Model

The total inventory holding cost over $[0, t]$ at facility 1 is

$$c_{w_1}\ W_1\ t = c_{w_1} \frac{\lambda(1 - p_1^{n_1})}{\mu_1(1 - p_1) - \lambda(1 - p_1^{n_1})}\ t.$$

The total inventory holding cost over $[0, t]$ at facility 2 is

$$c_{w_2}\ W_2\ t = c_{w_2} \frac{\lambda(1 - p_1^{n_1})(1 - p_2^{n_2})}{\mu_2(1 - p_2) - \lambda(1 - p_1^{n_1})(1 - p_2^{n_2})}\ t.$$

The total capacity expansion costs at facility 1 and facility 2 are

$$c_{m_1}\mu_1 \quad \text{and} \quad c_{m_2}\mu_2.$$

The total inventory holding cost of the batch demand orders over $[0, t]$ is

$$\frac{c_{w_3} D\ t}{2}.$$

Using the optimal raw material supply rate

$$\lambda = \frac{D}{(1 - p_1^{n_1})(1 - p_2^{n_2})\ t}$$

to simplify the inventory holding costs at both facility 1 and facility 2, we derive the overall cost,

$$\begin{aligned} & C(\mu_1, \mu_2, t) = c_{w_1}\ W_1\ t + c_{w_2}\ W_2\ t + c_{m_1}\mu_1 + c_{m_2}\mu_2 + \frac{c_{w_3} D\ t}{2} \\ = & \frac{c_{w_1}\lambda(1-p_1^{n_1})\ t}{\mu_1(1-p_1) - \lambda(1-p_1^{n_1})} + \frac{c_{w_2}\lambda(1-p_1^{n_1})(1-p_2^{n_2})\ t}{\mu_2(1-p_2) - \lambda(1-p_1^{n_1})(1-p_2^{n_2})} + c_{m_1}\mu_1 + c_{m_2}\mu_2 + \frac{c_{w_3} D\ t}{2} \\ = & \frac{c_{w_1}\ D}{\mu_1(1-p_1)(1-p_2^{n_2}) - \frac{D}{t}} + \frac{c_{w_2}\ D}{\mu_2(1-p_2) - \frac{D}{t}} + c_{m_1}\mu_1 + c_{m_2}\mu_2 + \frac{c_{w_3} D\ t}{2} \end{aligned}$$

Therefore the general optimization problem is

$$\begin{aligned} \text{minimize} \quad & C(\mu_1, \mu_2, t) = \frac{c_{w_1}\ D}{\mu_1(1-p_1)(1-p_2^{n_2}) - \frac{D}{t}} + \frac{c_{w_2}\ D}{\mu_2(1-p_2) - \frac{D}{t}} \\ & + c_{m_1}\mu_1 + c_{m_2}\mu_2 + \frac{c_{w_3} D\ t}{2} \\ \text{subject to} \quad & \mu_1 > \frac{D}{(1-p_1)(1-p_2^{n_2})\ t}; \\ & \mu_2 > \frac{D}{(1-p_2)\ t}; \\ & t \le t_0; \\ & \mu_1,\ \mu_2,\ t > 0. \end{aligned} \tag{4.2}$$

This is a nonlinear programming problem. Here $C(\mu_1, \mu_2, t)$ is a convex function. The verification of convexity can be carried out by deriving the Hessian of $C(\mu_1, \mu_2, t)$ and checking the second-order conditions (Luenberger [1984]). It is not difficult to check that all the first-order, second-order, and third-order principal minors are nonnegative. Similar discussion can be founded in Buzacott and Shanthikumar [1992]. To obtain the optimal solution, we only need to derive the stationary point of $C(\mu_1, \mu_2, t)$. The partial derivatives of $C(\mu_1, \mu_2, t)$ with respect to $\mu_1$, $\mu_2$, and $t$ are

$$\frac{\partial}{\partial \mu_1} C(\mu_1, \mu_2, t) = -\frac{c_{w_1}\ D\ (1-p_1)(1-p_2^{n_2})}{\left[\mu_1(1-p_1)(1-p_2^{n_2}) - \frac{D}{t}\right]^2} + c_{m_1},$$

$$\frac{\partial}{\partial \mu_2} C(\mu_1, \mu_2, t) = -\frac{c_{w_2}\ D\ (1-p_2)}{\left[\mu_2(1-p_2) - \frac{D}{t}\right]^2} + c_{m_2},$$

and

$$\frac{\partial}{\partial t} C(\mu_1, \mu_2, t) = -\frac{c_{w_1}\ D^2}{\left[\mu_1(1-p_1)(1-p_2^{n_2}) - \frac{D}{t}\right]^2\ t^2} - \frac{c_{w_2}\ D^2}{\left[\mu_2(1-p_2) - \frac{D}{t}\right]^2\ t^2} + \frac{c_{w_3}\ D}{2}.$$

We need to solve the following equations for $\mu_1$, $\mu_2$, and $t$,

$$\left\{ \begin{array}{lcl} \frac{\partial}{\partial \mu_1} C(\mu_1, \mu_2, t) & = & 0, \\ \frac{\partial}{\partial \mu_2} C(\mu_1, \mu_2, t) & = & 0, \\ \frac{\partial}{\partial t} C(\mu_1, \mu_2, t) & = & 0. \end{array} \right. \tag{4.3}$$

Solving $\frac{\partial}{\partial \mu_1} C(\mu_1, \mu_2, t) = 0$ for $\mu_1$, we have

$$\mu_1 = \sqrt{\frac{c_{w_1}\ D}{c_{m_1}(1-p_1)(1-p_2^{n_2})}} + \frac{D}{(1-p_1)(1-p_2^{n_2})\ t}. \tag{4.4}$$

Solving $\frac{\partial}{\partial \mu_2} C(\mu_1, \mu_2, t) = 0$ for $\mu_2$, we have

$$\mu_2 = \sqrt{\frac{c_{w_2}\ D}{c_{m_2}(1-p_2)}} + \frac{D}{(1-p_2)\ t}. \tag{4.5}$$

Replacing both $\mu_1$ and $\mu_2$ with (4.4) and (4.5) in terms of $t$ in $\frac{\partial}{\partial t} C(\mu_1, \mu_2, t)$ and simplifying it, we have

$$\frac{\partial}{\partial t} C(\mu_1, \mu_2, t) = -\frac{c_{m_1}\ D}{(1-p_1)(1-p_2^{n_2})\ t^2} - \frac{c_{m_2}\ D}{(1-p_2)\ t^2} + \frac{c_{w_3}\ D}{2}, \tag{4.6}$$

which is a function only depending on variable $t$. Solving $\frac{\partial}{\partial t} C(\mu_1, \mu_2, t) = 0$ for $t$, we have

$$t_c = \sqrt{\frac{2}{c_{w_3}} \left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]}. \tag{4.7}$$

From (4.6), we find that $\frac{\partial}{\partial t}C(\mu_1, \mu_2, t)$ is negative when $t \in (0, t_c)$ and is positive when $t \in (t_c, \infty)$. Therefore, $C(\mu_1, \mu_2, t)$ is decreasing when $t \in (0, t_c)$ and is increasing when $t \in (t_c, \infty)$. By the condition $t \in (0, t_0)$, where $t_0$ is the demand due date, the optimal $t$ of the problem (4.2) is

$$t^* = \min\{t_0, t_c\} = \min\left\{t_0, \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]}\right\}. \tag{4.8}$$

Therefore, the optimal analytical solution of the problem (4.2) is

$$\begin{cases} \mu_1^* = \sqrt{\frac{c_{w_1}\ D}{c_{m_1}(1-p_1)(1-p_2^{n_2})}} + \frac{D}{(1-p_1)(1-p_2^{n_2})\ t^*} \\ \mu_2^* = \sqrt{\frac{c_{w_2}\ D}{c_{m_2}(1-p_2)}} + \frac{D}{(1-p_2)\ t^*} \\ t^* = \min\left\{t_0,\ \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]}\right\}. \end{cases} \tag{4.9}$$

The optimal supplier rate (4.1) to facility 1

$$\lambda^* = \frac{D}{(1-p_1^{n_1})(1-p_2^{n_2})\ t^*}.$$

Both $\mu_1^*$ and $\mu_2^*$ are decreasing in time $t$ and increasing in demand $D$. $\mu_1^*$ is decreasing in $c_{m_1}$ and increasing in $c_{w_1}$, and $\mu_2^*$ is decreasing in $c_{m_2}$ and increasing in $c_{w_2}$. $t^*$ is decreasing in $c_{w_3}$ and increasing in $c_{m_1}$ and $c_{m_2}$. The optimal supplier rate $\lambda^*$ is decreasing in time $t$.

We now discuss the significance of optimal $t^*$. Here $t^*$ is a positive number, which is less than or equal to the demand due date $t_0$. Considering the planning issue, we can delay the start of the manufacturing activity by $t_0 - t^*$ time units. Using this strategy, we can save $c_{w_3}\ D\ (t_0 - t^*)$ inventory cost of demand orders.

The following discussion gives the corresponding optimal cost $C(\mu_1, \mu_2, t)$ of (4.2). The total optimal inventory cost at facility 1 is

$$\begin{aligned} \frac{c_{w_1}\ D}{\mu_1^*(1-p_1)(1-p_2^{n_2}) - \frac{D}{t^*}} &= \frac{c_{w_1}\ D}{\left[\sqrt{\frac{c_{w_1}\ D}{c_{m_1}(1-p_1)(1-p_2^{n_2})}} + \frac{D}{(1-p_1)(1-p_2^{n_2})\ t^*}\right](1-p_1)(1-p_2^{n_2}) - \frac{D}{t^*}} \\ &= \sqrt{\frac{c_{w_1}\ c_{m_1}\ D}{(1-p_1)(1-p_2^{n_2})}}. \end{aligned} \tag{4.10}$$

The total optimal inventory cost at facility 2 is

$$\begin{aligned}\frac{c_{w_2}\ D}{\mu_2^*(1-p_2)-\frac{D}{t^*}} &= \frac{c_{w_2}\ D}{\left[\sqrt{\frac{c_{w_2}\ D}{c_{m_2}(1-p_2)}}+\frac{D}{(1-p_2)\ t^*}\right](1-p_2)-\frac{D}{t^*}} \\ &= \sqrt{\frac{c_{w_2}\ c_{m_2}\ D}{1-p_2}}.\end{aligned} \tag{4.11}$$

The total optimal capacity expansion cost at both the facilities is

$$\begin{aligned}& c_{m_1}\mu_1^* + c_{m_2}\mu_2^* \\ = \ & c_{m_1}\left[\sqrt{\frac{c_{w_1}\ D}{c_{m_1}(1-p_1)(1-p_2^{n_2})}} + \frac{D}{(1-p_1)(1-p_2^{n_2})\ t^*}\right] + c_{m_2}\left[\sqrt{\frac{c_{w_2}\ D}{c_{m_2}(1-p_2)}} + \frac{D}{(1-p_2)\ t^*}\right] \\ = \ & \left[\sqrt{\frac{c_{w_1}\ c_{m_1}\ D}{(1-p_1)(1-p_2^{n_2})}} + \frac{c_{m_1}\ D}{(1-p_1)(1-p_2^{n_2})\ t^*}\right] + \left[\sqrt{\frac{c_{w_2}\ c_{m_2}\ D}{(1-p_2)}} + \frac{c_{m_2}\ D}{(1-p_2)\ t^*}\right] \\ = \ & \left[\sqrt{\frac{c_{w_1}\ c_{m_1}}{(1-p_1)(1-p_2^{n_2})}} + \sqrt{\frac{c_{w_2}\ c_{m_2}}{(1-p_2)}}\right]\sqrt{D} + \left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]\frac{D}{t^*}.\end{aligned} \tag{4.12}$$

The total inventory cost of batch demand orders

$$\frac{c_{w_3} D\ t^*}{2}. \tag{4.13}$$

Using results (4.10), (4.11), (4.12), and (4.13) to derive the total optimal costs $C(\mu_1^*, \mu_2^*, t^*)$ in (4.2) and simplifying it, we have

$$\begin{aligned}& C(\mu_1^*, \mu_2^*, t^*) \\ = \ & 2\left[\sqrt{\frac{c_{w_1}\ c_{m_1}}{(1-p_1)(1-p_2^{n_2})}} + \sqrt{\frac{c_{w_2}\ c_{m_2}}{1-p_2}}\right]\sqrt{D} + \left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]\frac{D}{t^*} + \frac{c_{w_3} D}{2}\ t^*.\end{aligned} \tag{4.14}$$

From (4.14), we find that the total optimal cost $C(\mu_1^*, \mu_2^*, t^*)$ is an increasing function in $c_{w_1}$, $c_{w_2}$, $c_{m_1}$, $c_{m_1}$, and $D$, and is a convex function in $t^*$.

## 5 Performance Bounds

Based on the analyses in Sections 2 and 4, we study the performance bounds for two different situations. First, we consider the case when actual demand exceeds the forecast by a given amount. Second, we consider the total cost for achieving a desired confidence level to meet the demand.

## 5.1 Demand Variation by Fixed Percentage

We assume that the total demand $D$ has an $\alpha$ ($\alpha \in [0,1]$) variation by the due date $t_0$. For example, $\alpha = 5\%$ means that demand $D$ has 5% variation. Under this situation, we need to consider the total demand $D_\alpha = D\ (1+\alpha)$ instead of $D$ for the worst case study. In the general optimization problem (4.2), we use $D_\alpha = D\ (1+\alpha)$ to replace $D$. Therefore, the optimal solution (4.9) becomes

$$\begin{cases} \mu_1^\alpha &= \sqrt{\frac{c_{w_1}\ D\ (1+\alpha)}{c_{m_1}(1-p_1)(1-p_2^{n_2})}} + \frac{D\ (1+\alpha)}{(1-p_1)(1-p_2^{n_2})\ t^*} \\ \mu_2^\alpha &= \sqrt{\frac{c_{w_2}\ D\ (1+\alpha)}{c_{m_2}(1-p_2)}} + \frac{D\ (1+\alpha)}{(1-p_2)\ t^*} \\ t^\alpha &= \min\left\{t_0,\ \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]}\right\}. \end{cases} \tag{5.1}$$

Here the optimal $t^\alpha$ is the same as $t^*$. The optimal supplier rate (4.1) to facility 1 is

$$\lambda^\alpha = \frac{D\ (1+\alpha)}{(1-p_1^{n_1})(1-p_2^{n_2})\ t^*}.$$

Thus, the optimal cost (4.14)

$$\begin{aligned} & C(\mu_1^\alpha, \mu_2^\alpha, t^\alpha) \\ = \ & 2\left[\sqrt{\frac{c_{w_1}\ c_{m_1}}{(1-p_1)(1-p_2^{n_2})}} + \sqrt{\frac{c_{w_2}\ c_{m_2}}{1-p_2}}\right]\sqrt{D\ (1+\alpha)} \\ & + \left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]\frac{D\ (1+\alpha)}{t^*} + \frac{c_{w_3} D\ (1+\alpha)}{2}\ t^*. \end{aligned} \tag{5.2}$$

Simplifying the difference in between (4.14) and (5.2), we derive the extra cost in the worst case

$$\begin{aligned} & C(\mu_1^\alpha, \mu_2^\alpha, t^\alpha) - C(\mu_1^*, \mu_2^*, t^*) \\ = \ & \alpha\ C(\mu_1^*, \mu_2^*, t^*) - 2\left[\sqrt{\frac{c_{w_1}\ c_{m_1}}{(1-p_1)(1-p_2^{n_2})}} + \sqrt{\frac{c_{w_2}\ c_{m_2}}{1-p_2}}\right]\sqrt{D\ (1+\alpha)}\left[\sqrt{1+\alpha} - 1\right]. \end{aligned} \tag{5.3}$$

Here $\alpha\ C(\mu_1^*, \mu_2^*, t^*)$ is the key part, and second part is negligible if $\alpha$ is small. In the worst case, for example $\alpha = 5\%$, the extra cost to order 5% more of the demand $D$ is roughly 5% more of the total optimal cost.

## 5.2 Confidence Level for Demand Satisfaction

In this section, we determine both $\lambda_d$ and $t$ for a given confidence level $\beta$ in worst case, such that

$$\mathrm{P}(N(t) \geq D) = 1 - \beta, \tag{5.4}$$

where $\{N(t),\ t \geq 0\}$ is a Poisson process with rate $\lambda_d$. For example, if $\beta = 5\%$, we want determine $\lambda_d$ and $t$ which guarantee 95% confidence that the total finished products by time $t$ will be more than the demand $D$.

From (5.4), we have

$$\mathrm{P}(N(t) < D) = \beta. \tag{5.5}$$

Since

$$\mathrm{P}(N(t) = k) = e^{-\lambda_d t} \frac{(\lambda_d t)^k}{k!}, \quad k = 0, 1, \cdots,$$

to solve (5.5) is equivalent to solve

$$\sum_{k=0}^{D-1} e^{-\lambda_d t} \frac{(\lambda_d t)^k}{k!} = \beta. \tag{5.6}$$

Let $D_\beta = \mathrm{E}N(t) = \lambda_d\, t$ and define

$$f(D_\beta) = \sum_{k=0}^{D-1} e^{-D_\beta} \frac{(D_\beta)^k}{k!} - \beta. \tag{5.7}$$

To solve (5.6) is equivalent to solve

$$f(D_\beta) = 0. \tag{5.8}$$

Instead of solving for $\lambda_d$ and $t$, we solve (5.8) for $D_\beta$. We consider the partial derivative of $f(D_\beta)$ with respect to $D_\beta$,

$$\frac{\partial}{\partial D_\beta} f(D_\beta) = -\sum_{k=0}^{D-1} e^{-D_\beta} \frac{(D_\beta)^k}{k!} + \sum_{k=0}^{D-1} e^{-D_\beta} \frac{(D_\beta)^{k-1}}{(k-1)!} = -e^{-D_\beta} \frac{(D_\beta)^{D-1}}{(D-1)!}, \tag{5.9}$$

which is negative. It indicates that $f(D_\beta)$ is a decreasing function in $D_\beta$ on $(0, \infty)$. Considering the two extreme cases, we have

$$\lim_{D_\beta \to 0^+} f(D_\beta) = 1 - \beta \quad \text{and} \quad \lim_{D_\beta \to \infty} f(D_\beta) = -\beta. \tag{5.10}$$

$f(D_\beta)$ is continuous in $D_\beta$, from both (5.9) and (5.10), we know that there is one and only one solution for (5.8). We cannot solve (5.8) nonlinear problem analytically. However, there are several numerical methods which can be used to solve this problem (5.8) (Atkinson [1978]). For this particular problem, we can use the bisection method (Atkinson [1978]). In practice, the interval $[D,\ 2D]$ can be selected as the initial search interval for the bisection method. The bisection method is computationally fast, and yet simple to implement, to solve the problem at hand.

We assume that $D_\beta^*$ is the unique solution of (5.8). In the general optimization problem (4.2), we use $D_\beta^*$ to replace $D$. The corresponding optimal solution (4.9) is

$$\left\{ \begin{array}{lcl} \mu_1^\beta & = & \sqrt{\frac{c_{w_1}\ D_\beta^*}{c_{m_1}(1-p_1)(1-p_2^{n_2})}} + \frac{D_\beta^*}{(1-p_1)(1-p_2^{n_2})\ t^*} \\ \mu_2^\beta & = & \sqrt{\frac{c_{w_2}\ D_\beta^*}{c_{m_2}(1-p_2)}} + \frac{D_\beta^*}{(1-p_2)\ t^*} \\ t^\beta & = & \min\left\{ t_0,\ \sqrt{\frac{2}{c_{w_3}} \left[ \frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2} \right]} \right\}. \end{array} \right. \tag{5.11}$$

The optimal $t^\alpha$ is identical to the $t^*$. The optimal rate of the Poisson process $\{N(t),\ t \geq 0\}$ is

$$\lambda_d^* = \frac{D_\beta^*}{t^*}.$$

The optimal raw material supply rate (4.1) to facility 1 is

$$\lambda^\alpha = \frac{D_\beta^*}{(1-p_1^{n_1})(1-p_2^{n_2})\ t^*}.$$

Thus, the optimal cost (4.14)

$$\begin{aligned} & C(\mu_1^\beta, \mu_2^\beta, t^\beta) \\ = \ & 2\left[\sqrt{\frac{c_{w_1}\ c_{m_1}}{(1-p_1)(1-p_2^{n_2})}} + \sqrt{\frac{c_{w_2}\ c_{m_2}}{1-p_2}}\right]\sqrt{D_\beta^*} + \left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]\frac{D_\beta^*}{t^*} + \frac{c_{w_3} D_\beta^*}{2}\ t^*. \end{aligned} \tag{5.12}$$

Practically, we can assume that

$$D < D_\beta^* < 2D.$$

Therefore, there exists an $\alpha \in [0,\ 1]$, such that

$$D_\beta^* = D\ (1+\alpha).$$

This is the same as the case discussed in Section 5.1. All discussions and results in Subsection 5.1 can be applied here, including the estimation of extra cost.

# 6 Sensitivity Analysis

We study the sensitivity of the total cost function with respect to the input data, such as the due date $t_0$, the demand $D$, the inventory costs $c_{w_1}$, $c_{w_2}$, and $c_{w_3}$, and the capacity expansion cost rates $c_{m_1}$ and $c_{m_1}$. The purpose of the sensitivity analysis is to determine how the optimal solution (4.9) and the total optimal cost (4.14) are affected by changes in the input data. This is important from a practical standpoint, since it the estimation of input data required for this model is often imprecise. Moreover, due to changing market conditions the supply chain planners have to often ascertain the impact of changes on the operating costs. We present the sensitivity analysis with respect the due date $t_0$ and the demand $D$ in this section. We discuss the effect on the objective function value $C(\mu_1^*, \mu_2^*, t^*)$ and the optimal solution $\mu_1^*$, $\mu_2^*$, and $t^*$ due to changes in the due date $t_0$ and the demand quantity $D$.

## 6.1 Due Date Changes

This section presents sensitivity analysis with respect the due date $t_0$. We denote that $\Delta t_0$ ($\Delta t_0 \in$ R) is the change in $t_0$. Replacing $t_0$ with $t_0 + \Delta t_0$ in (4.2), this general optimization problem is

$$
\begin{aligned}
\text{minimize} \quad & C(\mu_1, \mu_2, t) = \frac{c_{w_1} D}{\mu_1(1-p_1)(1-p_2^{n_2}) - \frac{D}{t}} + \frac{c_{w_2} D}{\mu_2(1-p_2) - \frac{D}{t}} \\
& \quad + c_{m_1}\mu_1 + c_{m_2}\mu_2 + \frac{c_{w_3} D\ t}{2} \\
\text{subject to} \quad & \mu_1 > \frac{D}{(1-p_1)(1-p_2^{n_2})\ t}; \\
& \mu_2 > \frac{D}{(1-p_2)\ t}; \\
& t \leq t_0 + \Delta t_0; \\
& \mu_1,\ \mu_2,\ t > 0.
\end{aligned}
\tag{6.1}
$$

The corresponding optimal solution (4.9) is

$$\begin{cases} \mu_1^{\Delta t_0} &= \sqrt{\frac{c_{w_1}\ D}{c_{m_1}(1-p_1)(1-p_2^{n_2})}} + \frac{D}{(1-p_1)(1-p_2^{n_2})\ t^{\Delta t_0}} \\ \mu_2^{\Delta t_0} &= \sqrt{\frac{c_{w_2}\ D}{c_{m_2}(1-p_2)}} + \frac{D}{(1-p_2)\ t^{\Delta t_0}} \\ t^{\Delta t_0} &= \min\left\{ t_0 + \Delta t_0,\ \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]}\right\}. \end{cases} \tag{6.2}$$

The optimal raw material supply rate (4.1) to facility 1 is

$$\lambda^{\Delta t_0} = \frac{D}{(1-p_1^{n_1})(1-p_2^{n_2})\ t^{\Delta t_0}}.$$

Thus, the optimal cost (4.14)

$$\begin{aligned} & C(\mu_1^{\Delta t_0}, \mu_2^{\Delta t_0}, t^{\Delta t_0}) \\ = \quad & 2\left[\sqrt{\frac{c_{w_1}\ c_{m_1}}{(1-p_1)(1-p_2^{n_2})}} + \sqrt{\frac{c_{w_2}\ c_{m_2}}{1-p_2}}\right]\sqrt{D} + \left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]\frac{D}{t^{\Delta t_0}} + \frac{c_{w_3} D}{2}\ t^{\Delta t_0}. \end{aligned} \tag{6.3}$$

From the special solution structure of $t^{\Delta t_0}$ in (6.2), the total optimal cost $C(\mu_1^*, \mu_2^*, t^*)$ is very sensitive to the change in due date $t_0$ when $t_0$ is small and is insensitive to due date variations when $t_0$ is large.

Based on particular values of $t_0 + \Delta t_0$, we study the sensitivity for the following four different situations.

**Case 1:** $t_0 \geq \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]}$ and $t_0 + \Delta t_0 \geq \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]}$

In this case,

$$t^{\Delta t_0} = \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]} = t^*.$$

Thus, the optimal solution and the total optimal cost are same as before. In other words, we have

$$\begin{cases} \mu_1^{\Delta t_0} &= \sqrt{\frac{c_{w_1} D}{c_{m_1}(1-p_1)(1-p_2^{n_2})}} + \frac{D}{(1-p_1)(1-p_2^{n_2})\ t^*} = \mu_1^* \\ \mu_2^{\Delta t_0} &= \sqrt{\frac{c_{w_2} D}{c_{m_2}(1-p_2)}} + \frac{D}{(1-p_2)\ t^*} = \mu_2^* \\ t^{\Delta t_0} &= \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]} = t^*, \end{cases}$$

$$\lambda^{\Delta t_0} = \frac{D}{(1-p_1^{n_1})(1-p_2^{n_2})\ t^*}\lambda^*,$$

and

$$\begin{aligned} & C(\mu_1^{\Delta t_0}, \mu_2^{\Delta t_0}, t^{\Delta t_0}) \\ = \ & 2\left[\sqrt{\frac{c_{w_1}\ c_{m_1}}{(1-p_1)(1-p_2^{n_2})}} + \sqrt{\frac{c_{w_2}\ c_{m_2}}{1-p_2}}\right]\sqrt{D} + \left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]\frac{D}{t^*} + \frac{c_{w_3} D}{2}\ t^*. \\ = \ & C(\mu_1^*, \mu_2^*, t^*) \end{aligned}$$

In this case the solution is insensitive to changes in $\Delta t_0$.

**Case 2:** $t_0 \geq \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]}$ and $t_0 + \Delta t_0 < \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]}$

In this case,

$$t^{\Delta t_0} = t_0 + \Delta t_0 < \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]} = t^*.$$

We have

$$\begin{cases} \mu_1^{\Delta t_0} &= \sqrt{\frac{c_{w_1} D}{c_{m_1}(1-p_1)(1-p_2^{n_2})}} + \frac{D}{(1-p_1)(1-p_2^{n_2})\ (t_0+\Delta t_0)} > \mu_1^* \\ \mu_2^{\Delta t_0} &= \sqrt{\frac{c_{w_2} D}{c_{m_2}(1-p_2)}} + \frac{D}{(1-p_2)\ (t_0+\Delta t_0)} > \mu_2^* \\ t^{\Delta t_0} &= t_0 + \Delta t_0 < t^*, \end{cases}$$

$$\lambda^{\Delta t_0} = \frac{D}{(1-p_1^{n_1})(1-p_2^{n_2})\ (t_0+\Delta t_0)} > \lambda^*$$

and

$$\begin{aligned} & C(\mu_1^{\Delta t_0}, \mu_2^{\Delta t_0}, t^{\Delta t_0}) \\ = \quad & 2\left[\sqrt{\tfrac{c_{w_1}\ c_{m_1}}{(1-p_1)(1-p_2^{n_2})}} + \sqrt{\tfrac{c_{w_2}\ c_{m_2}}{1-p_2}}\right]\sqrt{D} \\ & + \left[\tfrac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \tfrac{c_{m_2}}{1-p_2}\right]\tfrac{D}{t_0+\Delta t_0} + \tfrac{c_{w_3} D}{2}\ (t_0+\Delta t_0). \end{aligned}$$

Using result (4.7) and the follow up discussion, we have

$$C(\mu_1^{\Delta t_0}, \mu_2^{\Delta t_0}, t^{\Delta t_0}) \geq C(\mu_1^*, \mu_2^*, t^*).$$

**Case 3:** $t_0 < \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]}$ and $t_0 + \Delta t_0 \geq \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]}$

In this case,

$$t^{\Delta t_0} = t_0 + \Delta t_0 > t_0 = t^*.$$

We have

$$\begin{cases} \mu_1^{\Delta t_0} & = \sqrt{\frac{c_{w_1}\ D}{c_{m_1}(1-p_1)(1-p_2^{n_2})}} + \frac{D}{(1-p_1)(1-p_2^{n_2})\ (t_0+\Delta t_0)} < \mu_1^* \\ \mu_2^{\Delta t_0} & = \sqrt{\frac{c_{w_2}\ D}{c_{m_2}(1-p_2)}} + \frac{D}{(1-p_2)\ (t_0+\Delta t_0)} < \mu_2^* \\ t^{\Delta t_0} & = t_0 + \Delta t_0 > t^*, \end{cases}$$

$$\lambda^{\Delta t_0} = \frac{D}{(1-p_1^{n_1})(1-p_2^{n_2})\ (t_0+\Delta t_0)} < \lambda^*$$

and

$$\begin{aligned} & C(\mu_1^{\Delta t_0}, \mu_2^{\Delta t_0}, t^{\Delta t_0}) \\ = \quad & 2\left[\sqrt{\tfrac{c_{w_1}\ c_{m_1}}{(1-p_1)(1-p_2^{n_2})}} + \sqrt{\tfrac{c_{w_2}\ c_{m_2}}{1-p_2}}\right]\sqrt{D} \\ & + \left[\tfrac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \tfrac{c_{m_2}}{1-p_2}\right]\tfrac{D}{t_0+\Delta t_0} + \tfrac{c_{w_3} D}{2}\ (t_0+\Delta t_0). \end{aligned}$$

Using result (4.7) and the follow up discussion, we have

$$C(\mu_1^{\Delta t_0}, \mu_2^{\Delta t_0}, t^{\Delta t_0}) \le C(\mu_1^*, \mu_2^*, t^*).$$

The total optimal cost is reduced. This is always true when the due date $t_0$ is increased.

**Case 4:** $t_0 < \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]}$ and $t_0 + \Delta t_0 < \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]}$

In this case,

$$t^{\Delta t_0} = t_0 + \Delta t_0 \ge \quad \text{or} \quad \le t_0 = t^*.$$

We have

$$\begin{cases} \mu_1^{\Delta t_0} &= \sqrt{\frac{c_{w_1}\ D}{c_{m_1}(1-p_1)(1-p_2^{n_2})}} + \frac{D}{(1-p_1)(1-p_2^{n_2})\ (t_0+\Delta t_0)} \ge \quad \text{or} \quad \le \mu_1^* \\ \mu_2^{\Delta t_0} &= \sqrt{\frac{c_{w_2}\ D}{c_{m_2}(1-p_2)}} + \frac{D}{(1-p_2)\ (t_0+\Delta t_0)} \le \quad \text{or} \quad \ge \mu_2^* \\ t^{\Delta t_0} &= t_0 + \Delta t_0 \ge \quad \text{or} \quad \le t^*, \end{cases}$$

$$\lambda^{\Delta t_0} = \frac{D}{(1-p_1^{n_1})(1-p_2^{n_2})\ (t_0+\Delta t_0)} \le \quad \text{or} \quad \ge \lambda^*$$

and

$$\begin{aligned} & C(\mu_1^{\Delta t_0}, \mu_2^{\Delta t_0}, t^{\Delta t_0}) \\ = \ & 2\left[\sqrt{\frac{c_{w_1}\ c_{m_1}}{(1-p_1)(1-p_2^{n_2})}} + \sqrt{\frac{c_{w_2}\ c_{m_2}}{1-p_2}}\right]\sqrt{D} \\ & + \left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]\frac{D}{t_0+\Delta t_0} + \frac{c_{w_3} D}{2}\ (t_0+\Delta t_0). \end{aligned}$$

From result (4.7) and the follow up discussion, we have

$$C(\mu_1^{\Delta t_0}, \mu_2^{\Delta t_0}, t^{\Delta t_0}) \le \quad \text{or} \quad \ge C(\mu_1^*, \mu_2^*, t^*).$$

For all results here, we have two inequality signs. We take the first one if $\Delta t_0 \geq 0$ and take the second one otherwise. In other words, the total optimal overall cost is reduced if $\Delta t_0$ is positive and is increased otherwise.

## 6.2 Demand Changes

In this section, we conduct the sensitivity analysis with respect the demand $D$. We denote that $\Delta D$ ($\Delta D \in \mathrm{R}$) is the change of demand $D$. Replacing $D$ with $D + \Delta D$ in (4.2), the general optimization problem becomes

$$\begin{aligned} \text{minimize} \quad & C(\mu_1, \mu_2, t) = \frac{c_{w_1}\ (D+\Delta D)}{\mu_1(1-p_1)(1-p_2^{n_2}) - \frac{D+\Delta D}{t}} + \frac{c_{w_2}\ (D+\Delta D)}{\mu_2(1-p_2) - \frac{D+\Delta D}{t}} \\ & \quad + c_{m_1}\mu_1 + c_{m_2}\mu_2 + \frac{c_{w_3}(D+\Delta D)\ t}{2} \\ \text{subject to} \quad & \mu_1 > \frac{D+\Delta D}{(1-p_1)(1-p_2^{n_2})\ t}; \\ & \mu_2 > \frac{D+\Delta D}{(1-p_2)\ t}; \\ & t \leq t_0; \\ & \mu_1,\ \mu_2,\ t > 0. \end{aligned} \tag{6.4}$$

The corresponding optimal solution (4.9) is

$$\begin{cases} \mu_1^{\Delta D} = \sqrt{\frac{c_{w_1}\ (D+\Delta D)}{c_{m_1}(1-p_1)(1-p_2^{n_2})}} + \frac{D+\Delta D}{(1-p_1)(1-p_2^{n_2})\ t^{\Delta D}} \leq \ \text{or} \ \geq \mu_1^* \\ \mu_2^{\Delta D} = \sqrt{\frac{c_{w_2}\ (D+\Delta D)}{c_{m_2}(1-p_2)}} + \frac{D+\Delta D}{(1-p_2)\ t^{\Delta D}} \leq \ \text{or} \ \geq \mu_2^* \\ t^{\Delta D} = \min\left\{t_0,\ \sqrt{\frac{2}{c_{w_3}}\left[\frac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \frac{c_{m_2}}{1-p_2}\right]}\right\} = t^*. \end{cases} \tag{6.5}$$

The optimal raw material supply rate (4.1) to facility 1

$$\lambda^{\Delta D} = \frac{D + \Delta D}{(1-p_1^{n_1})(1-p_2^{n_2})\ t^{\Delta D}} \leq \ \text{or} \ \geq \lambda^*.$$

Thus, the optimal cost (4.14)

$$\begin{aligned} & C(\mu_1^{\Delta D}, \mu_2^{\Delta D}, t^{\Delta D}) \\ = \; & 2\left[\sqrt{\tfrac{c_{w_1}\ c_{m_1}}{(1-p_1)(1-p_2^{n_2})}} + \sqrt{\tfrac{c_{w_2}\ c_{m_2}}{1-p_2}}\right]\sqrt{D+\Delta D} \\ & + \left[\tfrac{c_{m_1}}{(1-p_1)(1-p_2^{n_2})} + \tfrac{c_{m_2}}{1-p_2}\right]\tfrac{D+\Delta D}{t^{\Delta D}} + \tfrac{c_{w_3}(D+\Delta D)}{2}\ t^{\Delta D}. \end{aligned} \tag{6.6}$$

Here $C(\mu_1^*, \mu_2^*, t^*)$ is a increasing function in demand $D$. We have

$$C(\mu_1^{\Delta D}, \mu_2^{\Delta D}, t^{\Delta D}) \leq \quad \text{or} \quad \geq C(\mu_1^*, \mu_2^*, t^*).$$

The optimal values of $\mu_1^*$, $\mu_2^*$, and $\lambda^*$ are increasing functions in demand $D$. Optimal $t^*$ is independent of demand $D$. All results except $t^{\Delta D}$ are true for the "$\leq$" relation if $\Delta D \leq 0$ and true for the "$\geq$" relation otherwise.

Further analysis can be conducted to ascertain the effect of changes in cost coefficients on the overall objective function.

# 7 Computational Results

Based on the masked data set from a major automobile manufacturer, we compute the optimal manufacturing rates, the raw material arrival rate, and the optimal total inventory and production costs. We conduct sensitivity analysis to determine the effects on the optimal solution (4.9) and the total optimal cost (4.14) when changes are made to the due date $t_0$ and demand quantity $D$.

For the purpose of the sensitivity analysis, we define the absolute relative ratio of change in the objective function value with respect to due date changes as

$$R^{\Delta t_0} = \left|\frac{C(\mu_1^*, \mu_2^*, t^*) - C(\mu_1^{\Delta t_0}, \mu_2^{\Delta t_0}, t^{\Delta t_0})}{C(\mu_1^*, \mu_2^*, t^*)}\right| \tag{7.1}$$

and the absolute relative ratio of change in the objective function value with respect to demand changes as

$$R^{\Delta D} = \left|\frac{C(\mu_1^*, \mu_2^*, t^*) - C(\mu_1^{\Delta D}, \mu_2^{\Delta D}, t^{\Delta D})}{C(\mu_1^*, \mu_2^*, t^*)}\right| \tag{7.2}$$

All computational results in this section is based on the parameters shown in Table 7.1. We present computational results for different values of due dates and batched demands.

Table 7.1: Input data.

| $p_1$ | $p_2$ | $n_1$ | $n_2$ | $c_{w_1}$ | $c_{w_2}$ | $c_{w_3}$ | $c_{m_1}$ | $c_{m_1}$ |
|---|---|---|---|---|---|---|---|---|
| 0.05 | 0.05 | 3 | 2 | 0.01 | 0.05 | 0.1 | 5 | 3 |

Table 7.2: Due date changes for $t_0 = 10$, $t_0 = 20$, and $D = 500$.

| $t_0$ | 10 | 10 | 10 | 10 | 20 | 20 | 20 | 20 |
|---|---|---|---|---|---|---|---|---|
| $D$ | 500 | 500 | 500 | 500 | 500 | 500 | 500 | 500 |
| $\Delta t_0$ | 5%$t_0$ | -5%$t_0$ | 10%$t_0$ | -10%$t_0$ | 5% $t_0$ | -5%$t_0$ | 10%$t_0$ | -10%$t_0$ |
| $\mu_1^*$ | 53.79 | 53.79 | 53.79 | 53.79 | 41.65 | 41.65 | 41.65 | 41.65 |
| $\mu_2^*$ | 55.59 | 55.59 | 55.59 | 55.59 | 43.49 | 43.49 | 43.49 | 43.49 |
| $t^*$ | 10.00 | 10.00 | 10.00 | 10.00 | 12.99 | 12.99 | 12.99 | 12.99 |
| $\lambda^*$ | 50.13 | 50.13 | 50.13 | 50.13 | 38.60 | 38.60 | 38.60 | 38.60 |
| $C(\mu_1^*, \mu_2^*, t^*)$ | 699.76 | 699.76 | 699.76 | 699.76 | 677.44 | 677.44 | 677.44 | 677.44 |
| $\mu_1^{\Delta t_0}$ | 53.53 | 54.06 | 53.27 | 54.32 | 41.65 | 41.65 | 41.65 | 41.65 |
| $\mu_2^{\Delta t_0}$ | 55.33 | 55.86 | 55.07 | 56.13 | 43.49 | 43.48 | 43.48 | 43.48 |
| $t^{\Delta t_0}$ | 10.05 | 9.95 | 10.10 | 9.90 | 12.99 | 12.99 | 12.99 | 12.99 |
| $\lambda^{\Delta t_0}$ | 49.88 | 50.38 | 49.64 | 50.64 | 38.60 | 38.60 | 38.60 | 38.60 |
| $C(\mu_1^{\Delta t_0}, \mu_2^{\Delta t_0}, t^{\Delta t_0})$ | 698.91 | 700.62 | 698.08 | 701.52 | 677.44 | 677.44 | 677.44 | 677.44 |
| $R^{\Delta t_0}$ | 0.0012 | 0.0012 | 0.0024 | 0.0025 | 0 | 0 | 0 | 0 |

From Tables 7.2 and 7.3, we can observe that the optimal manufacturing rates, the raw material arrival rate, and the optimal total inventory and production costs are sensitive with respect to the change of due date when $t_0$ is small and are insensitive when $t_0$ is large. The absolute relative ratio $R^{\Delta t_0}$ is not significant even when $t_0$ is small.

From Tables 7.4 and 7.5, we can observe that the optimal manufacturing rates, the raw material arrival rate, and the optimal total inventory and production costs are sensitive with respect to the change of demand $D$. The absolute relative ratio $R^{\Delta D}$ shows that the change in the total inventory and production costs is approximately proportional to the change in the batch demand quantities.

The computational results presented in Tables 7.2 – 7.5 are consistent with the theoretical results on sensitivity analysis presented in Section 6.

Table 7.3: Due date changes for $t_0 = 10$, $t_0 = 20$, and $D = 1000$.

| $t_0$ | 10 | 10 | 10 | 10 | 20 | 20 | 20 | 20 |
|---|---|---|---|---|---|---|---|---|
| $D$ | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 |
| $\Delta t_0$ | 5%$t_0$ | -5%$t_0$ | 10%$t_0$ | -10%$t_0$ | 5% $t_0$ | -5%$t_0$ | 10%$t_0$ | -10%$t_0$ |
| $\mu_1^*$ | 106.98 | 106.98 | 106.98 | 106.98 | 82.70 | 82.70 | 82.70 | 82.70 |
| $\mu_2^*$ | 109.45 | 109.45 | 109.45 | 109.45 | 85.24 | 85.24 | 85.24 | 85.24 |
| $t^*$ | 10.00 | 10.00 | 10.00 | 10.00 | 12.99 | 12.99 | 12.99 | 12.99 |
| $\lambda^*$ | 100.26 | 100.26 | 100.26 | 100.26 | 77.20 | 77.20 | 77.20 | 77.20 |
| $C(\mu_1^*, \mu_2^*, t^*)$ | 1383.08 | 1383.08 | 1383.08 | 1383.08 | 1338.45 | 1338.45 | 1338.45 | 1338.45 |
| $\mu_1^{\Delta t_0}$ | 106.46 | 107.51 | 105.94 | 108.05 | 82.70 | 82.70 | 82.70 | 82.70 |
| $\mu_2^{\Delta t_0}$ | 108.93 | 109.98 | 108.41 | 110.52 | 85.24 | 85.24 | 85.24 | 85.24 |
| $t^{\Delta t_0}$ | 10.05 | 9.95 | 10.10 | 9.90 | 12.99 | 12.99 | 12.99 | 12.99 |
| $\lambda^{\Delta t_0}$ | 99.76 | 100.77 | 99.27 | 101.28 | 77.20 | 77.20 | 77.20 | 77.20 |
| $C(\mu_1^{\Delta t_0}, \mu_2^{\Delta t_0}, t^{\Delta t_0})$ | 1381.39 | 1384.82 | 1379.73 | 1386.60 | 1338.45 | 1338.45 | 1338.45 | 1338.45 |
| $R^{\Delta t_0}$ | 0.0012 | 0.0013 | 0.0012 | 0.0013 | 0 | 0 | 0 | 0 |

Table 7.4: Demand changes for $t_0 = 10$, $t_0 = 20$, and $D = 500$.

| $t_0$ | 10 | 10 | 10 | 10 | 20 | 20 | 20 | 20 |
|---|---|---|---|---|---|---|---|---|
| $D$ | 500 | 500 | 500 | 500 | 500 | 500 | 500 | 500 |
| $\Delta D$ | 5%D | -5%D | 10%D | -10%D | 5%D | -5%D | 10%D | -10%D |
| $\mu_1^*$ | 53.79 | 53.79 | 53.79 | 53.79 | 41.65 | 41.65 | 41.65 | 41.65 |
| $\mu_2^*$ | 55.59 | 55.59 | 55.59 | 55.59 | 43.49 | 43.49 | 43.49 | 43.49 |
| $t^*$ | 10.00 | 10.00 | 10.00 | 10.00 | 12.99 | 12.99 | 12.99 | 12.99 |
| $\lambda^*$ | 50.13 | 50.13 | 50.13 | 50.13 | 38.60 | 38.60 | 38.60 | 38.60 |
| $C(\mu_1^*, \mu_2^*, t^*)$ | 699.76 | 699.76 | 699.76 | 699.76 | 677.44 | 677.44 | 677.44 | 677.44 |
| $\mu_1^{\Delta D}$ | 56.45 | 51.13 | 59.12 | 48.46 | 43.71 | 39.60 | 45.77 | 37.54 |
| $\mu_2^{\Delta D}$ | 58.30 | 52.89 | 61.00 | 50.18 | 45.58 | 41.38 | 47.68 | 39.28 |
| $t^{\Delta D}$ | 10.00 | 10.00 | 10.00 | 10.00 | 12.99 | 12.99 | 12.99 | 12.99 |
| $\lambda^{\Delta D}$ | 52.64 | 47.63 | 55.14 | 45.12 | 40.53 | 36.67 | 42.46 | 34.74 |
| $C(\mu_1^{\Delta D}, \mu_2^{\Delta D}, t^{\Delta D})$ | 734.03 | 665.46 | 768.30 | 631.15 | 710.60 | 644.26 | 743.75 | 611.06 |
| $R^{\Delta D}$ | 0.049 | 0.049 | 0.098 | 0.098 | 0.049 | 0.049 | 0.098 | 0.098 |

Table 7.5: Demand changes for $t_0 = 10$, $t_0 = 20$, and $D = 1000$.

| $t_0$ | 10 | 10 | 10 | 10 | 20 | 20 | 20 | 20 |
|---|---|---|---|---|---|---|---|---|
| $D$ | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 |
| $\Delta D$ | 5%D | -5%D | 10%D | -10%D | 5%D | -5%D | 10%D | -10%D |
| $\mu_1^*$ | 106.98 | 106.98 | 106.98 | 106.98 | 82.70 | 82.70 | 82.70 | 82.70 |
| $\mu_2^*$ | 109.45 | 109.45 | 109.45 | 109.45 | 85.24 | 85.24 | 85.24 | 85.24 |
| $t^*$ | 10.00 | 10.00 | 10.00 | 10.00 | 12.99 | 12.99 | 12.99 | 12.99 |
| $\lambda^*$ | 100.26 | 100.26 | 100.26 | 100.26 | 77.20 | 77.20 | 77.20 | 77.20 |
| $C(\mu_1^*, \mu_2^*, t^*)$ | 1383.08 | 1383.08 | 1383.08 | 1383.08 | 1338.45 | 1338.45 | 1338.45 | 1338.45 |
| $\mu_1^{\Delta D}$ | 112.29 | 101.67 | 117.60 | 96.35 | 86.80 | 78.60 | 90.90 | 74.50 |
| $\mu_2^{\Delta D}$ | 114.82 | 104.08 | 120.18 | 98.71 | 89.39 | 81.08 | 93.55 | 76.92 |
| $t^{\Delta D}$ | 10.00 | 10.00 | 10.00 | 10.00 | 12.99 | 12.99 | 12.99 | 12.99 |
| $\lambda^{\Delta D}$ | 105.28 | 95.25 | 110.29 | 90.24 | 81.06 | 73.34 | 84.92 | 69.48 |
| $C(\mu_1^{\Delta D}, \mu_2^{\Delta D}, t^{\Delta D})$ | 1451.23 | 1314.91 | 1519.36 | 1246.71 | 1404.37 | 1272.50 | 1470.26 | 1206.53 |
| $R^{\Delta D}$ | 0.049 | 0.049 | 0.098 | 0.098 | 0.049 | 0.049 | 0.098 | 0.098 |

# 8 Summary and Conclusions

This paper presented a general model of two facilities in tandem with feedbacks, which produce a single final product. Many stochastic models (Veatch and Wein [1994], Walrand [1988], and Bowman [1994]) discuss the situations when the rework trips for unacceptable products are unlimited or the manufacturing quality is perfect without any feedback. We derive the total expected work-in-process inventory. A total cost model is introduced as a nonlinear programming problem. We derive the optimal manufacturing rates, the raw material arrival rate, and the optimal total inventory and production costs. We also discuss the sensitivity of the optimal control policy with respect to the demand variations and due date changes.

The problem studied in this paper is of importance to several real world supply chain applications. The model considered in this paper can be easily extended to a serial supply chain with arbitrarily many processing facilities. The formulation can also be extended to include multiple final product configurations, being manufactured in the same facilities, such as the different automobile configurations based on color, drive train and engine. The opportunity cost of quality (Bowman [1994]) can be used in conjunction with the results of this paper to formulate quality control strategies.

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## Biographical Sketches

**Jin Wang** is an Associate Professor in the Department of Mathematics and Computer Science at Valdosta State University. He received his Ph.D. degree in Industrial Engineering from Purdue University. His research interests include stochastic modeling and optimization, queueing methods for services and manufacturing, computational finance, and Monte Carlo simulation. He is a member of IIE and INFORMS.

**Abhijit V. Deshmukh** is an Assistant Professor in the Department of Mechanical and Industrial Engineering at the University of Massachusetts, Amherst. He received a B.E. in Production Engineering from the University of Bombay, an M.S. in Industrial Engineering from SUNY at Buffalo, and a Ph.D. in Industrial Engineering from Purdue University. His research interests are in design and control of manufacturing systems. He is a member of IIE, INFORMS and ASME.

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