Mean-Variance-VaR Based Portfolio Optimization

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Abstract

We propose two new portfolio optimization approaches. The first is a two-stage portfolio optimization approach using both mean-variance and mean-VaR approaches in a priority order. The second is a general mean-variance-VaR approach using both variance and VaR as a double-risk measure simultaneously. Our approaches overcome the shortcomings of both mean-variance and mean-VaR approaches while providing additional strengths and flexibility. Comparing the mean-variance method with the mean-VaR method, we derive two general results: (1) the mean-variance efficient set is not a subset of mean-VaR efficient set, and vice versa; (2) the mean-variance equivalent set is not a mean-VaR equivalent set, and vice versa. We find that in general the results of the multivariate normality case can not be extended to the non-normality case.

KEY WORDS: Mean-VaR Analysis; Mean-Variance Analysis; Risk Management; Portfolio Optimization

1 Introduction

The mean-variance approach is the earliest method to solve the portfolio selection problem (Markowitz [1952, 1959]). The principle of diversification is the foundation of this method and it still has wide application in risk management. However, there are some arguments against it though this approach has been accepted and appreciated by practitioners and academics for a number of years

(Korn [1997]). The variance of the portfolio return is the only risk measure of this method. Controlling (minimizing) the variance does not only lead to low deviation from the expected return on the down side, but also on the up side. It may bound the possible gains too.

In recent years, VaR has become a new benchmark for managing and control risk (Dowd [1998], Jorion [1997], RiskMetrics^{TM} [1995]). Unfortunately, VaR based risk management has two shortcomings. First, VaR measures have difficulties aggregating individual risks, and sometimes discourage diversification (Artzner et al [1998]). Second, the VaR based risk management is only focusing on controlling the probability of loss, rather than its magnitude (Basak and Shapiro [1999]). The expected losses, conditional on the states where there are large losses, may be higher sometimes.

The mean-variance approach encourages risk diversification, but the mean-VaR approach discourages risk diversification sometimes. The mean-variance approach does not only control the risk of return on the down side, but also bounds the possible gain on the up side while the mean-VaR approach only controls risk of return on the down side. Another limitation of both approaches is that the underlying distribution of the rate of return is not well understood, and there are no higher degree information is utilized except means, covariances (variances), or values of VaR.

In this paper, we propose a two-stage portfolio optimization approach which has all the strengths of the mean-VaR and the mean-variance approaches, and overcomes their shortcomings as the two stages complement one another. This approach also uses more information of the underlying distribution of the portfolio return. In this approach, variance and VaR as risk measures are used separately in two stages according to a priority order of the two risk measures. In stage one, we use a primary risk measure to collect all efficient portfolios. In stage two, we use a secondary risk measure to re-evaluate (optimize) these efficient portfolios from stage one. This approach provides better results than the mean-variance and the mean-VaR approaches considered separately.

Instead of using one single risk measure, we also propose a general mean-variance-VaR approach using variance and VaR as a double-risk measure simultaneously. The mean-variance-VaR efficient portfolio may not be mean-variance efficient or mean-VaR efficient. We also show that the mean-variance and the mean-VaR approaches are special cases of the mean-variance-VaR approach.

Many papers have been published that are related to this work, the most related being works by Alexander and Baptista [2000] and Basak and Shapiro [1999]. The first study compares the mean-variance and mean-VaR approaches for two special cases: multivariate normal distribution and multivariate t-distribution. The second study analyzes optimal policies focusing on the VaR based risk management. They propose a new risk measure to control both conditional expected loss and VaR. Our work does not only compare the mean-variance and mean-VaR approaches in a general case, but also merges the two approaches into one single approach. As an extension of the two-stage optimization approach, we propose a different approach, which also controls both conditional expected loss and VaR.

The rest of this paper is organized into six sections. In Section 2, we review the mean-variance approach. In Section 3, we review the concept of VaR and the mean-VaR approach. In Section 4, we compare the mean-variance approach with the mean-VaR for a general case without any assumption for the distribution of portfolio return. In Section 5, we propose a new a approach of portfolio selection: a two-stage portfolio optimization strategy. In Section 6, we propose a more general portfolio optimization strategy: the mean-variance-VaR model. The usual mean-variance model and the mean-VaR model are special situations of this model. Both variance and VaR are used as the risk measures during the procedure of optimization. In Section 7, we summarize our results and discuss some possible future research problems.

2 Mean-Variance Approach

In this section, we briefly review the mean-variance approach.

Suppose that there are n securities with rates of return X_i $(i = 1, \cdots, n)$. The means and covariances of these rates of return are,

$$\mu_i = \mathrm{E}(X_i) \quad \text{and} \quad \sigma_{ij} = \mathrm{Cov}(X_i, X_j), \quad i, j = 1, \cdots, n.$$

The portfolio vector is

$$w = (w_1, \cdots, w_n)' \in \mathrm{R}^n \quad \text{and} \quad \sum_{i=1}^{n} w_i = 1.$$

We define that set W is a collection of all possible portfolios:

$$W = \left\{ w \in R^n \;\middle|\; \sum_{i=1}^{n} w_i = 1 \right\}$$

The total return of portfolio w is

$$R_w = \sum_{i=1}^{n} w_i X_i.$$

Its mean and variance are

$$\mu_w = \mathrm{E}(R_w) = \mathrm{E}\left(\sum_{i=1}^{n} w_i X_i\right) = \sum_{i=1}^{n} w_i \mu_i$$

and

$$\sigma_w^2 = \mathrm{Var}\left(\sum_{i=1}^{n} w_i X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}.$$

There are two common models that utilize the mean-variance principle. The idea of the first model is that for a given upper bound σ_0^2 for the variance of the portfolio return, select a portfolio w, such that μ_w is maximum with $\sigma_w^2 \le \sigma_0^2$:

$$\begin{array}{cl} \max & \mu_w \\ w \in W & \\ \text{s.t.} & \sigma_w^2 \le \sigma_0^2. \end{array} \tag{2.1}$$

The second model states that for a given lower bound μ_0 for the mean of the portfolio return, select a portfolio w, such that σ_w^2 is minimum with $\mu_w \ge \mu_0$:

$$\begin{array}{cl} \min & \sigma_w^2 \\ w \in W & \\ \text{s.t.} & \mu_w \ge \mu_0. \end{array} \tag{2.2}$$

3 Mean-VaR Approach

In this section, we briefly review the concept of VaR and the mean-VaR approach.

The VaR measures the worst expected loss over a given time interval under normal market conditions at a given confidence level, and provides users a summary measure of market risk. Precisely, the VaR at the $100(1-\alpha)\%$ confidence level of a portfolio w for a specific time period is the rate of return q_w such that the probability of that portfolio having a rate of return of $-q_w$ or less is α:

$$\mathrm{P}\left(R_w \le -q_w\right) = \alpha. \tag{3.1}$$

Here $-q_w$ is also called the α^{th} quantile of the distribution of R_w. Similar to the mean-variance method, we define two models for the mean-VaR principle. The first one is that for a given upper bound q_0 for the VaR of the portfolio return, select a portfolio w, such that μ_w is the maximum with $q_w \leq q_0$:

$$\begin{array}{ll} \max & \mu_w \\ w \in W & \\ \text{s.t.} & q_w \leq q_0. \end{array} \tag{3.2}$$

The second model states that for a given lower bound μ_0 for the mean of the portfolio return, select a portfolio w, such that its VaR q_w is minimum with $\mu_w \geq \mu_0$:

$$\begin{array}{ll} \min & q_w \\ w \in W & \\ \text{s.t.} & \mu_w \geq \mu_0. \end{array} \tag{3.3}$$

4 Comparison of Mean-Variance and Mean-VaR Approaches

In this section, we compare the mean-VaR approach with the mean-variance approach. The two approaches are using completely different risk measures to optimize portfolios. The mean-variance approach only uses of the mean and variance of portfolio return. The Mean-VaR approach only uses the mean and VaR of the portfolio return. Both approaches have many advantages; however they do not sufficiently use the information from the distribution of the portfolio return. As risk measures, variance and VaR are independent in general. One exception is that the VaR measure is proportional (linear) to the variance measure (standard deviation) in the multivariate normal case. Example 4.1 shows that a mean-variance efficient portfolio is not a mean-VaR efficient portfolio. Conversely, and Example 4.2 shows that a mean-VaR efficient portfolio is not a mean-variance efficient portfolio. Examples 4.3 and 4.4 show that mean-variance equivalent and mean-VaR equivalent are excluding each other through two particular examples.

Example 4.1 *A mean-variance efficient portfolio is not a mean-VaR efficient portfolio.*

We consider a simple two-security portfolio selection problem. The rate of return for the first security is

$$X_1 = Z,$$

where Z is the standard normal $N(0,1)$ with mean, variance, and VaR,

$$\mu_{X_1} = 0, \quad \sigma^2_{X_1} = 1, \quad \text{and} \quad q_{X_1} = z_\alpha,$$

where $1-\alpha$ is the confidence level (say $\alpha = 0.05$), and $-z_\alpha$ is the α^{th} quantile of the standard normal distribution, such that

$$\alpha = \int_{-\infty}^{-z_\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

The rate of return for the second security is

$$X_2 = 2Z + 2z_\alpha$$

with mean, variance, and VaR,

$$\mu_{X_2} = 2z_\alpha \quad \text{and} \quad \sigma^2_{X_2} = 4, \quad \text{and} \quad q_{X_2} = 0.$$

The correlation of X_1 and X_2 is

$$\mathrm{Corr}(X_1, X_2) = \mathrm{Corr}(Z, 2Z + 2z_\alpha) = 1.$$

For any portfolio w, the variance of its return $R_w = w_1 X_1 + w_2 X_2$ is

$$\mathrm{Var}(R_w) = \mathrm{Var}(w_1 X_1 + w_2 X_2) = (w_1 + 2w_2)^2 = (2 - w_1)^2.$$

This variance reaches minimum value 1 when $w_1 = 1$. Therefore $w^* = (1,0)$ is a mean-variance efficient portfolio with mean, variance, and VaR

$$\mu_{w^*} = 0, \quad \sigma^2_{w^*} = 1, \quad \text{and} \quad q_{w^*} = z_\alpha.$$

But this portfolio is not mean-VaR efficient. Consider portfolio $w^{**} = (0,1)$. Both mean and VaR of $R_{w^{**}} = X_2$ are better,

$$\mu_{w^{**}} = \mu_{X_2} = 2z_\alpha > 0 = \mu_{w^*} \quad \text{and} \quad q_{w^{**}} = q_{X_2} = 0 < z_\alpha = q_{w^*}.$$

Example 4.2 *A mean-VaR efficient portfolio is not a mean-variance efficient portfolio.*

We construct a two-security portfolio selection example. The rate of return for the first security is

$$X_1 = Z,$$

where Z is the standard normal$N(0,1)$ with mean, variance, and VaR,

$$\mu_{X_1} = 0, \quad \sigma^2_{X_1} = 1, \quad \text{and} \quad q_{X_1} = z_\alpha,$$

where z_α is the α^{th} quantile of the standard normal distribution. The rate of return for the second security is from the Normal-mixture family (Lehmann [1983]). In general, the Normal-mixture random variable is distributed with probability p as $N(\xi, \delta^2)$ and probability $1-p$ as $N(\eta, \tau^2)$. Its cumulative distribution function (cdf) is then given by

$$\mathrm{F}(x) = p\Phi\left(\frac{x-\xi}{\delta}\right) + (1-p)\Phi\left(\frac{x-\eta}{\tau}\right),$$

where Φ is the cdf of standard normal $N(0,1)$. Its density is

$$f(x) = \frac{p}{\sqrt{2\pi}\ \delta} e^{-\frac{(x-\xi)^2}{2\delta^2}} + \frac{1-p}{\sqrt{2\pi}\ \tau} e^{-\frac{(x-\eta)^2}{2\tau^2}}.$$

This model is widely used in financial industry. For examples, Clark [1973] uses it to fit cotton futures price data, Duffie and Pan [1997] use it to simulate fat tailed distributions, and Hull and White [1998] use it calculate VaR when daily changes in market variables are not normally distributed. We pick the following parameters for the X_2:

$$p = \frac{1}{2}, \quad \xi = 0, \quad \eta = 0, \quad \delta = \frac{1}{2}, \quad \text{and} \quad \tau = \sqrt{2}.$$

Therefore its cdf and density are

$$\mathrm{F}_{X_2}(x) = \frac{1}{2}\Phi(2x) + \frac{1}{2}\Phi\left(\frac{x}{\sqrt{2}}\right)$$

and

$$f_{X_2}(x) = \frac{1}{\sqrt{2\pi}}\left[e^{-2x^2} + \frac{1}{\sqrt{8}} e^{-\frac{x^2}{4}}\right].$$

with mean

$$\mu_{X_2} = p\xi + (1-p)\eta = 0,$$

and variance

$$\sigma^2_{X_2} = p\delta^2 + (1-p)\tau^2 = \frac{9}{8}.$$

At the 91.69% confidence level ($\alpha = 0.0831$), both X_1 and X_2 are sharing the same VaR value:

$$q_{X_1} = q_{X_2} = 1.381.$$

Figure 4.1 shows the detail. We construct the strong positive correlation between X_1 and X_2 in the following way. We let U be the uniform random variable over interval (0,1) and define

$$X_1 = \mathrm{F}_{X_1}^{-1}(U) \quad \text{and} \quad X_2 = \mathrm{F}_{X_2}^{-1}(U),$$

where $\mathrm{F}_{X_1}^{-1}$ and $\mathrm{F}_{X_2}^{-1}$ are inverse functions of F_{X_1} and F_{X_2}, respectively. It can be shown easily and theoretically that both X_1 and X_2 have the desired distributions with strong positive correlation. Based on the special setting, for any portfolio w at $\alpha = 0.0831$, the mean and VaR of $R_w = w_1X_1 + w_2X_2$ are constants:

$$\mu_w = 0, \quad \text{and} \quad q_w = 1.381.$$

Therefore portfolio $w^* = (0,1)$ is mean-VaR efficient (here $R_{w^*} = X_2$). But it is not mean-variance efficient. Consider portfolio $w^{**} = (1,0)$ (here $R_{w^{**}} = X_1$). We have

$$\mu_{w^{**}} = \mu_{X_1} = 0 = \mu_{X_2} = \mu_{w^*}, \quad q_{w^{**}} = q_{X_1} = 1.381 = q_{X_2} = q_{w^*},$$

and

$$\sigma^2_{w^{**}} = \sigma^2_{X_1} = 1 < \frac{9}{8} = \sigma^2_{X_2} = \sigma^2_{w^*}.$$

Portfolio w^{**} has a better (smaller) variance the portfolio w^*.

Summarizing Example 4.1 and Example 4.2, we have the following result.

Proposition 4.1 *In general, a mean-variance efficient set is not a subset of the mean-VaR efficient set and vice versa.*

Example 4.3 *Two mean-variance equivalent portfolios are not mean-VaR equivalent.*

We demonstrate an example to show that two mean-variance equivalent portfolios are not mean-VaR equivalent. We consider an example similar to Example 4.2 with different parameters. We pick $X_1 = Z$ is the standard normal and X_2 is a Normal-Mixture with parameters

$$p = \frac{1}{2}, \quad \xi = 0, \quad \eta = 0, \quad \delta = \frac{1}{2}, \quad \text{and} \quad \tau = \frac{\sqrt{7}}{2}.$$

Then its mean and variance are

$$\mu_{X_2} = p\xi + (1-p)\eta = 0,$$

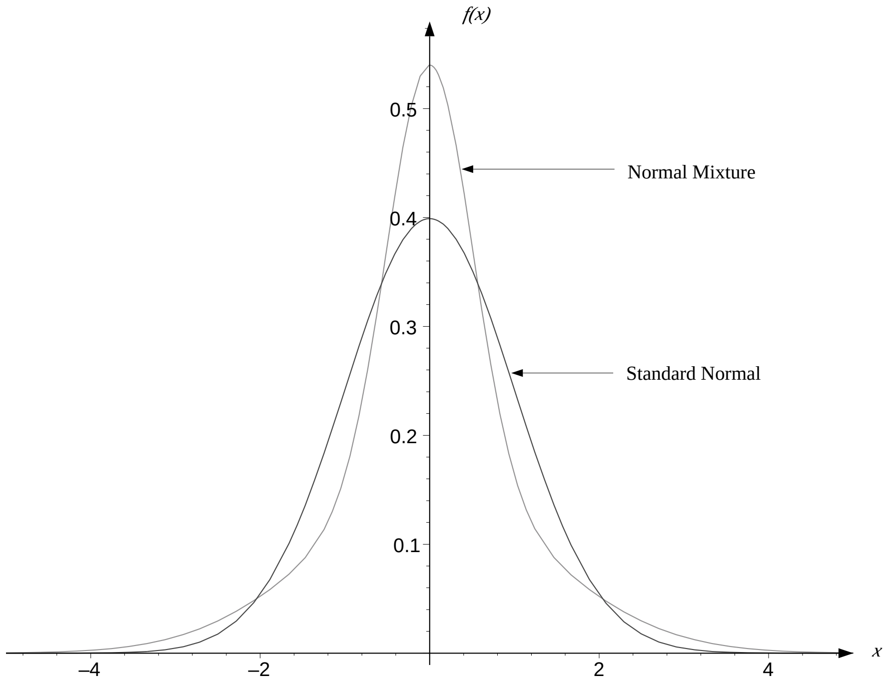

Figure 4.1: Probability Density Distributions of Standard Normal and Normal-Mixture with the same mean, VaR (at $\alpha = 0.0831$), and different variance values.

and

$$\sigma^2_{X_2} = p\delta^2 + (1-p)\tau^2 = 1.$$

Therefore X_1 and X_2 have the same mean and variance and they are mean-variance equivalent portfolios; however, they have different VaR values. See Figure (4.2) for details. Considering two portfolios $w^* = (1, 0)$ and $w^{**} = (0, 1)$, we have

$$R_{w^*} = X_1 \quad \text{and} \quad R_{w^{**}} = X_2.$$

The two portfolios are mean-variance equivalent since they have the same means and variances. But they are not mean-VaR equivalent. For example, they have different VaR values at $\alpha = 0.01$,

$$q_{w^*} = q_{X_1} = 3.09 \quad \text{and} \quad q_{w^{**}} = q_{X_2} = 2.717.$$

Example 4.4 *Mean-VaR equivalent portfolios are not mean-variance equivalent.*

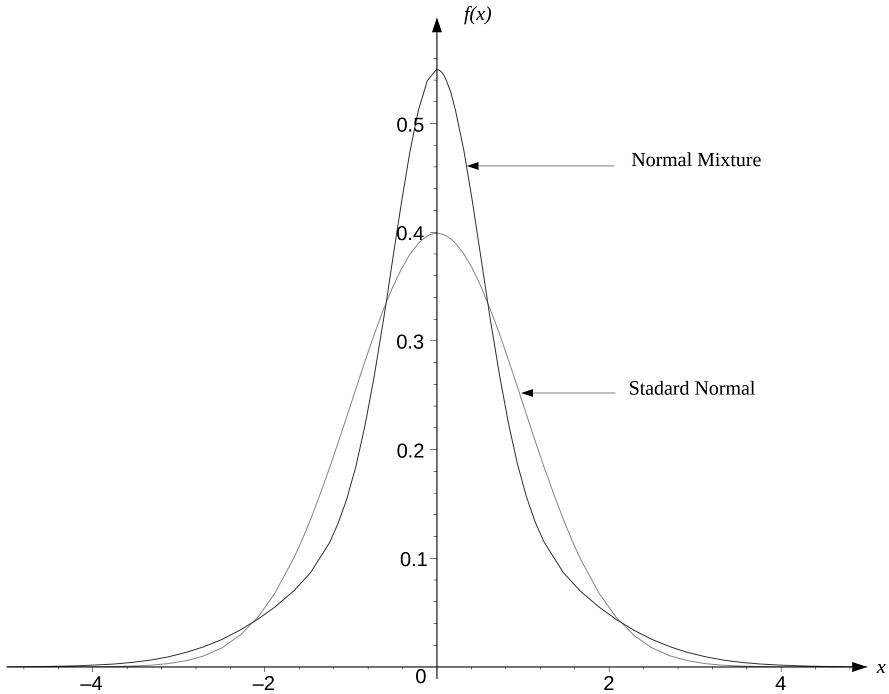

Figure 4.2: Probability Density Distributions of Standard Normal and Normal-Mixture with the same mean, variance, and different VaR values (at $\alpha = 0.01$).

We use Example 4.2 again to show that two mean-VaR equivalent portfolios are not mean-variance equivalent. For portfolios $w^* = (1, 0)$ and $w^{**} = (0, 1)$ at $\alpha = 0.0831$, we have

$$\mu_{w^*} = \mu_{X_1} = 0 = \mu_{X_2} = \mu_{w^{**}}, \quad \text{and} \quad q_{w^*} = q_{X_1} = 1.381 = q_{X_2} = q_{w^{**}}.$$

The two portfolios are mean-VaR equivalent. But they have different variances:

$$\sigma^2_{w^*} = \sigma^2_{X_1} = 1 < \frac{9}{8} = \sigma^2_{X_2} = \sigma^2_{w^{**}}.$$

Summarizing Example 4.3 and Example 4.4, we have the following result.

Proposition 4.2 *In general, a mean-variance equivalent set is not a mean-VaR equivalent set and vice versa.*

Example 4.5 *Multivariate Normal Case*

Under the normality assumption, the portfolio return R_w is a $N(\mu_w, \sigma^2_w)$ random variable with VaR,

$$q_w = z_\alpha \sigma_w - \mu_w. \tag{4.1}$$

If w^* is a mean-VaR efficient portfolio, then for any portfolio, we have

$$q_w \geq q_{w^*} \quad \text{if} \quad \mu_w \geq \mu_{w^*}.$$

Using result (4.1), we have

$$\sigma_w = \frac{1}{z_\alpha}(q_w + \mu_w) \geq \frac{1}{z_\alpha}(q_{w^*} + \mu_{w^*}) = \sigma_{w^*}.$$

Therefore we have the following result.

Lemma 4.1 *Under the normality assumption, a mean-VaR efficient portfolio is mean-variance efficient.*

This result shows that under normality assumption, the mean-VaR efficient frontier is a subset of the mean-variance efficient frontier. From Example 4.1, we know that the mean-variance efficient portfolio is not mean-VaR efficient. Summarizing the above discussion, we have the following result.

Proposition 4.3 *Under the normality assumption, the mean-VaR efficient frontier is a proper subset of the mean-variance efficient frontier.*

This result is consistent with the result of Alexander and Baptista [2000]. Equation (4.1) describes the linear relationship among mean, standard deviation, and VaR. For any two portfolios, if two of their three parameters are equal, then their third parameters must be equal. We summarize this as the following result.

Proposition 4.4 *Under the normality assumption, mean-variance equivalent portfolios are mean-VaR equivalent and vice versa.*

In this section, we have derived results for both normal and non-normal situations. These results are different. In practice, the normality assumption is heavily used for modeling non-normal situations. Sometimes, we may derive good approximations using the normality assumption for non-normal situations; however, it is not the case here in this section. Thus, we should be very careful when we apply or generalize these results derived under the normality assumption to a general non-normal situation.

5 Two-Stage Optimization Approach

In this section, we propose a two-stage portfolio optimization approach. It incorporates the strengths of both mean-VaR and mean-variance approaches, and as well as overcomes their shortcomings as the two strategies complement each other. In this approach, variance and VaR are used as risk measures in the two stages separately according to a priority order of risk measures. In stage one, we collect all efficient portfolios based on a primary risk measure. In stage two, we re-evaluate (optimize) these efficient portfolios from stage one based on a secondary risk measure. Based on the priority order of risk measures, we propose several versions of the two-stage portfolio optimization models.

5.1 Mean-Variance Model with Minimal VaR

If variance is the primary risk measure of the portfolio return, In the first stage, we collect all mean-variance efficient portfolios. Then we re-evaluate (optimize) these efficient portfolios using VaR as a risk measure of the portfolio return in second stage. we propose two models.

- **Min-Max Model**:

$$\min_{w \in W_{opt}} \quad q_w \tag{5.1}$$

where W_{opt} is a solution set of

$$\begin{aligned} \max_{w \in W} \quad & \mu_w \\ \text{s.t.} \quad & \sigma_w^2 \leq \sigma_0^2. \end{aligned}$$

- **Min-Min Model**:

$$\min_{w \in W_{opt}} \quad q_w \tag{5.2}$$

where W_{opt} is a solution set of

$$\begin{aligned} \min_{w \in W} \quad & \sigma_w^2 \\ \text{s.t.} \quad & \mu_w \geq \mu_0. \end{aligned}$$

Under the normality assumption, from equation (4.1) we see that any two portfolios that are mean-variance efficient have the same means, variances, and VaR values. In this situation, the second stage of the optimization is not needed.

5.2 Mean-VaR Model with Minimal Variance

In this subsection, we propose two models in which we assume that VaR is the primary portfolio risk measure. In the first stage, we collect all mean-VaR efficient portfolios. Then in the second stage, we re-evaluate (optimize) these efficient portfolios using variance as a secondary risk measure.

- **Min-Max Model**:

$$\min_{w \in W_{opt}} \quad \sigma_w^2 \tag{5.3}$$

where W_{opt} is a solution set of

$$\begin{aligned} \max_{w \in W} \quad & \mu_w \\ \text{s.t.} \quad & q_w \le q_0. \end{aligned}$$

- **Min-Min Model**:

$$\min_{w \in W_{opt}} \quad \sigma_w^2 \tag{5.4}$$

where W_{opt} is a solution set of

$$\begin{aligned} \min_{w \in W} \quad & q_w \\ \text{s.t.} \quad & \mu_w \ge \mu_0. \end{aligned}$$

In Example 4.2, both portfolios w^* ($R_{w^*} = X_2$) and w^{**} ($R_{w^{**}} = X_1$) are mean-VaR efficient portfolios at $\alpha = 0.083$, since they have the same mean and VaR values,

$$\mu_{w^*} = \mu_{w^{**}} = 0 \quad \text{and} \quad q_{w^*} = q_{w^{**}} = 1.381.$$

But under the two-stage optimization strategy, portfolio w^{**} is better than portfolio w^* since w^{**} has a smaller variance ($\sigma^2_{w^{**}} = 1$) than portfolio w^* ($\sigma^2_{w^*} = 9/8$). Once again, both models (5.3) and (5.4) are the same as the mean-VaR model under the normality assumption. Therefore there is no need to have the second stage optimization.

5.3 Mean-VaR Model with Minimal Conditional Expected Loss

As we mentioned in Section 3, the mean-VaR approach is only focusing on controlling the probability of loss, rather than its magnitude. Basak and Shapiro [1999] propose a new risk measure to overcome this shortcoming. They define

$$\mathrm{E}\left(-R_w I_{\{R_w \leq -q_w\}}\right) \leq \epsilon,$$

where $-R_w$ is the loss, and $\epsilon \geq 0$ is a constant. This constraint penalizes both a high probability of loss, and a high expected loss given there is a loss. As an alternative and extension of the two-stage optimization method, we propose two models which control both conditional expected loss and VaR. We assume that the VaR is the primary portfolio risk measure, and a new risk measure, the conditional expected loss, is the secondary portfolio risk measure.

- **Min-Max Model**:

$$\begin{array}{ll} \min\limits_{w \in W_{opt}} & \mathrm{E}\left(-R_w | R_w \leq -q_w\right) \end{array} \tag{5.5}$$

where W_{opt} is a solution set of

$$\begin{array}{ll} \max\limits_{w \in W} & \mu_w \\ \text{s.t.} & q_w \leq q_0. \end{array}$$

- **Min-Min Model**:

$$\begin{array}{ll} \min\limits_{w \in W_{opt}} & \mathrm{E}\left(-R_w | R_w \leq -q_w\right) \end{array} \tag{5.6}$$

where W_{opt} is a solution set of

$$\begin{array}{ll} \min\limits_{w \in W} & q_w \\ \text{s.t.} & \mu_w \geq \mu_0. \end{array}$$

In Example 4.2, both portfolios w^* ($R_{w^*} = X_2$) and w^{**} ($R_{w^{**}} = X_1$) are mean-VaR efficient portfolios at $\alpha = 0.083$, since they have the same mean and VaR values,

$$\mu_{w^*} = \mu_{w^{**}} = 0 \quad \text{and} \quad q_{w^*} = q_{w^{**}} = 1.381.$$

But under the two-stage optimization strategy, portfolio w^{**} is better than portfolio w^* since

$$\mathrm{E}\left(-R_{w^{**}} | R_{w^{**}} \leq -q_{w^{**}}\right) = \mathrm{E}\left(-X_1 | X_1 \leq -q_{X_1}\right) = 0.111$$

and

$$\mathrm{E}\left(-R_{w^{*}} | R_{w^{*}} \leq -q_{w^{*}}\right) = \mathrm{E}\left(-X_2 | X_2 \leq -q_{X_2}\right) = 0.128.$$

This result is consistent with the result in Subsection 5.2. At some degree in the second stage, we also can reduce the conditional expected loss if we use variance as the risk measure.

6 Mean-Variance-VaR Approach

In this section, we propose a general mean-variance-VaR model for portfolio optimization with two variations. We use both variance and VaR as risk control measures. Our models cover both the mean-variance model and the mean-VaR model. In other words, the two models are special cases of our models.

The first model is that for given upper bounds σ_0^2 and q_0 for the variance and VaR of the portfolio return respectively, select a portfolio w, such that μ_w is the maximum with $\sigma_w^2 \leq \sigma_0^2$ and $q_w \leq q_0$:

$$\begin{array}{cl} \max & \mu_w \\ w \in W & \\ \text{s.t.} & \sigma_w^2 \leq \sigma_0^2; \\ & q_w \leq q_0. \end{array} \tag{6.1}$$

Comparing with the mean-variance model or the mean-VaR model, we use double-risk measures instead of one single risk measure. The mean-variance-VaR efficient portfolio may not be mean-variance efficient or mean-VaR efficient. Moreover, the mean-variance model (2.1) and the mean-VaR model (3.2) are special cases of our model (6.1):

- When $q_0 = \infty$, our model (6.1) becomes the mean-variance model (2.1);
- When $\sigma_0^2 = \infty$, our model (6.1) becomes the mean-VaR model (3.2).

The second model is that for a given lower bound μ_0^2 for the mean of the portfolio return, select a portfolio w, such that the convex combination of variance and VaR of the portfolio return

$\beta\sigma_w^2 + (1-\beta)q_w$ is the minimum with $\mu_w \geq \mu_0$:

$$\begin{array}{cl} \min & \beta\sigma_w^2 + (1-\beta)q_w \\ w \in W & \\ \text{s.t.} & \mu_w \geq \mu_0. \end{array} \tag{6.2}$$

Here $\beta \in [0,1]$ is an agent defined constant. For the two extreme values of β, we have

- When $\beta = 1$, our model (6.2) becomes the mean-variance model (2.2);
- When $\beta = 0$, our model (6.2) becomes the mean-VaR model (3.3).

How do we select the constant β? This is an open problem. The answer depends partly on the agent's understanding of the two risk measures. For example, if the variance is more important than the VaR, put more weight on variance, i.e., select $\beta > 0.5$. Otherwise, select $\beta \leq 0.5$. Under the normality assumption, the VaR is a linear function of the standard deviation: $q_w = z_\alpha\sigma_w - \mu_w$. This suggests that we may need to use the same risk scale for the variance and VaR in the model, at least for the normal case. Possible alternatives to the objective function of model (6.2) are

$$\beta\sigma_w^2 + (1-\beta)q_w^2 \quad \text{and} \quad \beta\sigma_w + (1-\beta)q_w.$$

From the computational point view, $\beta\sigma_w^2+(1-\beta)q_w^2$ is better than $\beta\sigma_w+(1-\beta)q_w$ since square-root takes more computation time than square. We also can substitute the objective function of model (6.2) by a general utility function $f(\sigma_w^2, q_w)$.

7 Conclusions

In this paper we have discussed and compared the mean-variance approach with the mean-VaR approach. We find that in general the two approaches are different in two ways: (1) the mean-variance efficient set is not a subset of mean-VaR efficient set, and vice versa; and (2) the mean-variance equivalent set is not a mean-VaR equivalent set, and vice versa. But under the normality assumption, the mean-VaR efficient frontier is a proper subset of the mean-variance efficient frontier, and Mean-variance equivalent portfolios are mean-VaR equivalent and vice versa. Results derived in this paper under the normality and non-normality assumptions are totally different. This suggests to us that we should be very careful when we apply or generalize these results derived under the normality assumption to a general non-normal situation. The two-stage portfolio optimization approach is a combination of the mean-variance and the mean-VaR approaches. It maintains the strengths while overcoming the shortcomings of both approaches. The mean-variance-VaR approach uses variance and VaR as a double-risk measure simultaneously. The mean-variance and the mean-VaR approaches are special cases of this approach. The two new approaches we proposed are better than and improve both mean-variance and the mean-VaR approaches. The open questions are: (1) how to determine the optimal constant β in the general mean-variance-VaR model; and (2) how to generalize to the continuous-time portfolio situation.

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