

Chapter 3 Some Special Distributions

Section 3.4 The Normal Distribution

Consider the integral $I = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$. This integral exists since $e^{-\frac{y^2}{2}}$ is continuous and

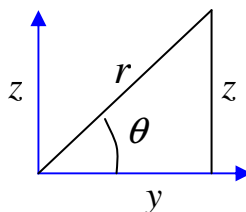
bounded by $0 < e^{-\frac{y^2}{2}} < e^{-|y|+1}$, for $-\infty < y < \infty$ and $\int_{-\infty}^{\infty} e^{-|y|+1} dy = 2e$.

To evaluate the integral I consider the following work. Note, that $I > 0$ and

$$I \times I = I^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \right) \left(\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \right) \Rightarrow I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{y^2+z^2}{2}} dydz .$$
 We evaluate this

integral by changing to polar coordinates.

$$\sin \theta = \frac{z}{r} \text{ and } \cos \theta = \frac{y}{r}$$



Let $y = r \cos \theta$ and $z = r \sin \theta$. The Jacobian of the transformation is given by

$$J = \begin{vmatrix} \frac{dy}{dr} & \frac{dy}{d\theta} \\ \frac{dz}{dr} & \frac{dz}{d\theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$J = r \cos^2 \theta - (-r \sin^2 \theta) = r \cos^2 \theta + r \sin^2 \theta = r(\cos^2 \theta + \sin^2 \theta) = r$$

$$\text{Now, } I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-\frac{(r \cos \theta)^2 + (r \sin \theta)^2}{2}} |J| dr d\theta = \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}} r dr d\theta = \int_0^{2\pi} \int_0^{\infty} e^{-u} du d\theta$$

$$I^2 = \int_0^{2\pi} 1 d\theta = 2\pi \Rightarrow I = \pm \sqrt{2\pi} \quad \text{Since } I > 0 \text{ we have } I = \sqrt{2\pi} .$$

$$I = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = \sqrt{2\pi} \Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1 .$$

Now, introduce a new variable, say $Y = \frac{X-a}{b}$; $\Rightarrow dy = \frac{1}{b} dx$ where $b > 0$.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{\left(\frac{x-a}{b}\right)^2}{2}} dx = 1$$

$f(x) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{\left(\frac{x-a}{b}\right)^2}{2}}$; where $-\infty < x < \infty$ Note: $f(x)$ satisfies the conditions of being a pdf of the continuous type random variable. It's called the Normal distribution function and it's denoted by $X \sim N(\mu, \sigma^2)$ or $X \sim N(\mu, \sigma)$.

Moment generating function

$$M(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{b\sqrt{2\pi}} e^{-\frac{\left(\frac{x-a}{b}\right)^2}{2}} dx = \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2 + tx} dx = \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{tx - \frac{x^2 - 2ax + a^2}{2b^2}} dx$$

$$M(t) = \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{\frac{-2b^2tx + x^2 - 2ax + a^2}{2b^2}} dx = \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{x^2 - (2b^2t + 2a)x + a^2}{2b^2}} dx$$

Now complete the square
 $\left(\frac{2b^2t + 2a}{2}\right)^2 = (b^2t + a)^2$

$$M(t) = \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{x^2 - (2b^2t + 2a)x + (b^2t + a)^2 - (b^2t + a)^2 + a^2}{2b^2}} dx = \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{\left(x - (b^2t + a)\right)^2 - (b^2t + a)^2 + a^2}{2b^2}} dx$$

$$M(t) = e^{\frac{(b^2t + a)^2 - a^2}{2b^2}} \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-\frac{\left(x - (b^2t + a)\right)^2}{2b^2}} dx = e^{\frac{(b^2t + a)^2 - a^2}{2b^2}} = e^{\frac{(b^2t + a - a)(b^2t + a + a)}{2b^2}}$$

$$M(t) = e^{\frac{(b^2t)(b^2t + a + a)}{2b^2}} = e^{\frac{b^4t^2 + 2ab^2t}{2b^2}} = e^{\frac{b^2t^2}{2} + at}$$

Hence, $M(t) = e^{\frac{b^2t^2}{2} + at}$.

Now, we would like to find a and b . Work for the derivative, $M'(t)$. Let

$$y = e^{\frac{b^2 t^2}{2} + at} \quad \text{and} \quad u = \frac{b^2 t^2}{2} + at \quad \text{then} \quad y = e^u.$$

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = (e^u) \cdot (a + b^2 t) = \left(e^{\frac{b^2 t^2}{2} + at} \right) (a + b^2 t) = M(t)(a + b^2 t).$$

$$M'(t) = M(t)(a + b^2 t) \Big|_{t=0} = a. \quad \text{Since, } \mu = M'(0) = a \Rightarrow a = \mu.$$

$$M''(t) = M(t)(b^2) + M(t)(a + b^2 t)^2 \Big|_{t=0} = a^2 + b^2.$$

$$\text{Since, } \sigma^2 = M''(0) - (M'(0))^2 = a^2 + b^2 - a^2 = b^2 \Rightarrow b^2 = \sigma^2.$$

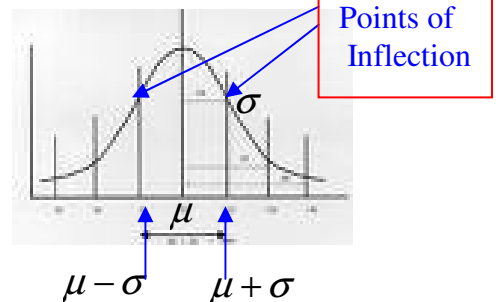
Now, the probability density function, pdf, and the mgf can be written in terms of μ and σ .

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \quad \text{where } -\infty < x < \infty \quad \text{and} \quad M(t) = e^{\frac{\sigma^2 t^2}{2} + \mu t}.$$

If $X \sim N(\mu=0, \sigma^2=1)$ then $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$; where $-\infty < x < \infty$ and it's called the standard normal distribution.

Properties of the Normal distribution:

1. Symmetric about μ .
2. Has the maximum at $x = \mu$ and is $\frac{1}{\sigma\sqrt{2\pi}}$.
3. The points at $x = \mu \pm \sigma$ are points of inflection.



Theorem 1: If the random variable $X \sim N(\mu, \sigma^2)$, then the random variable $W = \frac{X-\mu}{\sigma}$ has a normal distribution with $\mu=0$ and $\sigma=1$; i.e $W \sim N(0, 1)$.

Proof: $G(W) = P(W \leq w) = P\left(\frac{X-\mu}{\sigma} \leq w\right) = P(X \leq \sigma w + \mu) = \int_{-\infty}^{\sigma w + \mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$

$$g(w) = G'(W) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\sigma w + \mu - \mu}{\sigma}\right)^2} \cdot \frac{d}{dw}(\sigma w + \mu) - 0 = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(w)^2} \cdot \sigma = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}$$

Hence, $g(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}$; $-\infty < w < \infty$ and $W \sim N(0, 1)$.

If the random variable $X \sim N(\mu, \sigma^2)$, then $P(X = c_1) = 0$ since X is a continuous random variable.

$$P(c_1 \leq X \leq c_2) = P(X \leq c_2) - P(X \leq c_1) = P\left(W \leq \frac{c_2 - \mu}{\sigma}\right) - P\left(W \leq \frac{c_1 - \mu}{\sigma}\right)$$

Example: If $X \sim N(2, 25)$, compute $P(0 \leq X \leq 10)$.

$$P(0 \leq X \leq 10) = P(X \leq 10) - P(X \leq 0) = P\left(W \leq \frac{10-2}{5}\right) - P\left(W \leq \frac{0-2}{5}\right)$$

$$P(0 \leq X \leq 10) = P(W \leq 1.6) - P(W \leq -0.4) = 0.945 - 0.345 = 0.60$$

Example: Compute $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.

$$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = P\left(\frac{\mu - 2\sigma - \mu}{\sigma} \leq W \leq \frac{\mu + 2\sigma - \mu}{\sigma}\right) = P(-2 \leq W \leq 2)$$

$$= P(W \leq 2) - P(W \leq -2) = 0.977 - .023 = 0.954$$

Example: Suppose that 10 percent of the probability distribution that is $N(\mu, \sigma^2)$ is below 60 and that 5 percent is above 90. Find μ and σ .

We are going to need two equations since we have two unknowns.

$$1. P(X \leq 60) = 0.10 \Rightarrow P\left(W \leq \frac{60-\mu}{\sigma}\right) = 0.10 \Rightarrow \frac{60-\mu}{\sigma} = -1.282 \text{ from the table.}$$

$$2. P(X \geq 90) = 0.05 \Rightarrow P\left(W \leq \frac{90-\mu}{\sigma}\right) = 0.95 \Rightarrow \frac{90-\mu}{\sigma} = 1.645 \text{ from the table.}$$

Now, solving for μ and σ , we found that $\mu = 73.1$ and $\sigma = 10.2$.

Theorem 2: If the random variable $X \sim N(\mu, \sigma^2)$, then the random variable

$V = \left(\frac{X-\mu}{\sigma}\right)^2$ has a Chi-square distribution with $r = 1$ degrees of freedom ; i.e $V \sim \chi^2_{(r=1)}$.

Proof: Note that $V = W^2$ implying that $V > 0$ and where $W = \frac{X-\mu}{\sigma}$; i.e $W \sim N(0, 1)$.

$G(V) = P(V \leq v) = P(W^2 \leq v) = P(-\sqrt{v} \leq W \leq \sqrt{v}) = 2P(0 \leq W \leq \sqrt{v})$ since the

function is even. So, $G(V) = 2 \int_0^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw ; V > 0$

$$g(v) = G'(V) = 2 \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{v})^2}{2}} \frac{d}{dv}(\sqrt{v}) - 0 = \frac{2}{\sqrt{2\pi}} e^{-\frac{v}{2}} \cdot \frac{1}{2\sqrt{v}} = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{v}} e^{-\frac{v}{2}} = \frac{1}{\sqrt{\pi} 2^{\frac{1}{2}}} v^{\frac{1}{2}-1} e^{-\frac{v}{2}}$$

Since $g(v)$ is a pdf ; i.e $\int_0^{\infty} \frac{1}{\sqrt{\pi} 2^{\frac{1}{2}}} v^{\frac{1}{2}-1} e^{-\frac{v}{2}} dv = 1$ implies that $\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right)$.

Hence, $g(v) = \frac{1}{\Gamma\left(\frac{1}{2}\right) 2^{\frac{1}{2}}} v^{\frac{1}{2}-1} e^{-\frac{v}{2}} ; 0 < v < \infty$; i.e $V \sim \chi^2_{(r=1)}$.

Homework: 3.4 : 5, 8, 10, 12 and 13 pp. 168-169 .