

## Chapter 3 Some Special Distributions

### Section 3.3 The Gamma and Chi-Square Distributions

**Gamma:** From calculus we know that  $\int_0^{\infty} y^{\alpha-1} e^{-y} dy$  exists for  $\alpha > 0$  and that the value of the integral is a positive number.  $\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy$  is called the Gamma function.

If  $\alpha = 1$  then  $\Gamma(\alpha = 1) = \int_0^{\infty} y^{1-1} e^{-y} dy = \int_0^{\infty} e^{-y} dy = 1$ .

Integration By Parts

If  $\alpha > 1$  then  $\Gamma(\alpha) = (\alpha - 1) \int_0^{\infty} y^{\alpha-2} e^{-y} dy = (\alpha - 1) \int_0^{\infty} y^{(\alpha-1)-1} e^{-y} dy = (\alpha - 1)\Gamma(\alpha - 1)$ .

Accordingly, if  $\alpha$  is a positive integer greater than 1, then  $\Gamma(\alpha) = (\alpha - 1)(\alpha - 2)(\alpha - 3) \dots (3)(2)(1)\Gamma(1) = (\alpha - 1)!$

Note: Since  $\Gamma(1) = 1$ , suggests that  $0! = 1$ .

Now consider the Gamma function,  $\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy$ , and the change of variable

$Y = \frac{x}{\beta}$ ; where  $\beta > 0$ .

$$\Gamma(\alpha) = \int_0^{\infty} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}} \frac{d}{dx}\left(\frac{x}{\beta}\right) dx = \int_0^{\infty} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}} \frac{1}{\beta} dx = \frac{1}{\beta^{\alpha}} \int_0^{\infty} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$$

Now divide both sides by  $\Gamma(\alpha)$ . We have  $1 = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$ . Since,

$\alpha > 0$ ,  $\beta > 0$ , and  $\Gamma(\alpha) > 0$ , we have a new probability density function (pdf) called the Gamma function,  $f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$ ;  $0 < x < \infty$ . So, the random variable  $X$  has a

Gamma function with parameters  $\alpha$  and  $\beta$  and it's denoted by  $X \sim \text{Gamma}(\alpha, \beta)$ .

Find the Moment Generating Function, mgf, of the Gamma function.

$$M(t) = E(e^{tx}) = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{tx} x^{\alpha-1} e^{-\frac{x}{\beta}} dx = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x(\frac{1}{\beta}-t)} dx = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x(\frac{1-\beta t}{\beta})} dx$$

$$M(t) = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x(\frac{1-\beta t}{\beta})} dx = \left(\frac{1}{\beta^\alpha}\right) \left(\frac{\beta}{1-\beta t}\right)^\alpha \int_0^{\infty} \frac{1}{\Gamma(\alpha) \left(\frac{\beta}{1-\beta t}\right)^\alpha} x^{\alpha-1} e^{-x(\frac{1-\beta t}{\beta})} dx = \left(\frac{1}{\beta^\alpha}\right) \left(\frac{\beta}{1-\beta t}\right)^\alpha$$

$$M(t) = \frac{1}{(1-\beta t)^\alpha} \text{ or } (1-\beta t)^{-\alpha} \text{ where } (1-\beta t) > 0 \Rightarrow t < \frac{1}{\beta}.$$

Mean and Variance of the Gamma function

$$M'(t) = -\alpha (1-\beta t)^{-\alpha-1} (-\beta) \text{ and } M'(0) = -\alpha (1-\beta \cdot 0)^{-\alpha-1} (-\beta) = \alpha\beta \Rightarrow \mu = \alpha\beta.$$

$$M''(t) = \alpha\beta \left[ -(\alpha+1)(1-\beta t)^{-\alpha-2} (-\beta) \right] \text{ and}$$

$$M''(0) = \alpha\beta \left[ -(\alpha+1)(1-\beta \cdot 0)^{-\alpha-2} (-\beta) \right] = \alpha(\alpha+1)\beta^2.$$

$$\sigma^2 = M''(0) - (M'(0))^2 = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2 \Rightarrow \sigma^2 = \alpha\beta^2.$$

Chi-Square Distribution

Now, let  $\alpha = \frac{r}{2}$  where  $r$  is a positive integer and  $\beta = 2$ .  $f(x) = \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{2}} ; 0 < x < \infty$ .

$$M(t) = \frac{1}{(1-2t)^{\frac{r}{2}}} \text{ or } (1-2t)^{-\frac{r}{2}} \text{ where } t < \frac{1}{2}.$$

$X$  has a Chi-square distribution with  $r$  degrees of freedom and it's denoted by  $X \sim \chi_{(r)}^2$ .

$$\mu = \left(\frac{r}{2}\right)2 = r \text{ and } \sigma^2 = \left(\frac{r}{2}\right)2^2 = 2r$$

Examples 3.3.5 and 3.3.6 on pp. 153-154.

Example 3.3.6. Consider the Gamma function where  $X \sim \text{Gamma}\left(\frac{r}{2}, \beta\right)$ ;

$$f(x) = \frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{\beta}} dx; 0 < x < \infty. \text{ If } Y = \frac{2X}{\beta}, \text{ find the pdf of } Y.$$

$$\text{Start from } G(Y) = P(Y \leq y) = P\left(\frac{2X}{\beta} \leq y\right) = P\left(X \leq \frac{\beta y}{2}\right) = \int_0^{\frac{\beta y}{2}} \frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{\beta}} dx$$

$$g(y) = G'(Y) = \frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}} \left(\frac{\beta y}{2}\right)^{\frac{r}{2}-1} e^{-\frac{\beta y}{2} \cdot \frac{1}{\beta}} \cdot \frac{d}{dy}\left(\frac{\beta y}{2}\right) - 0 = \frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}} \left(\frac{\beta y}{2}\right)^{\frac{r}{2}-1} e^{-\frac{\beta y}{2} \cdot \frac{1}{\beta}} \cdot \left(\frac{\beta}{2}\right)$$

$$g(y) = \frac{1}{\Gamma\left(\frac{r}{2}\right)2^{\frac{r}{2}}} y^{\frac{r}{2}-1} e^{-\frac{y}{2}}; 0 < y < \infty. \text{ Hence, } Y \sim \chi_{(r)}^2$$

**Homework: Section 3.3 : 1, 2, 3, 7, 16 pp. 157-158**

**Note: Problem 3.3.16 should say “Find the cdf of  $Y = -2\ln(X)$ .”**