

Chapter 3 Some Special Distributions

Section 3.3 The Gamma and Chi-Square Distributions

Gamma: From calculus we know that $\int_0^{\infty} y^{\alpha-1} e^{-y} dy$ exists for $\alpha > 0$ and that the value of the integral is a positive number. $\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy$ is called the Gamma function.

If $\alpha = 1$ then $\Gamma(\alpha = 1) = \int_0^{\infty} y^{1-1} e^{-y} dy = \int_0^{\infty} e^{-y} dy = 1$.

Integration By Parts

If $\alpha > 1$ then $\Gamma(\alpha) = (\alpha - 1) \int_0^{\infty} y^{\alpha-2} e^{-y} dy = (\alpha - 1) \int_0^{\infty} y^{(\alpha-1)-1} e^{-y} dy = (\alpha - 1)\Gamma(\alpha - 1)$.

Accordingly, if α is a positive integer greater than 1, then $\Gamma(\alpha) = (\alpha - 1)(\alpha - 2)(\alpha - 3) \dots (3)(2)(1)\Gamma(1) = (\alpha - 1)!$

Note: Since $\Gamma(1) = 1$, suggests that $0! = 1$.

Now consider the Gamma function, $\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy$, and the change of variable $Y = \frac{x}{\beta}$; where $\beta > 0$.

$$\Gamma(\alpha) = \int_0^{\infty} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}} \frac{d}{dx}\left(\frac{x}{\beta}\right) dx = \int_0^{\infty} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}} \frac{1}{\beta} dx = \frac{1}{\beta^{\alpha}} \int_0^{\infty} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$$

Now divide both sides by $\Gamma(\alpha)$. We have $1 = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$. Since,

$\alpha > 0$, $\beta > 0$, and $\Gamma(\alpha) > 0$, we have a new probability density function (pdf) called the Gamma function, $f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$; $0 < x < \infty$. So, the random variable X has a

Gamma function with parameters α and β and it's denoted by $X \sim \text{Gamma}(\alpha, \beta)$.

Find the Moment Generating Function, mgf, of the Gamma function.

$$M(t) = E(e^{tx}) = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{tx} x^{\alpha-1} e^{-\frac{x}{\beta}} dx = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x(\frac{1}{\beta}-t)} dx = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x(\frac{1-\beta t}{\beta})} dx$$

$$M(t) = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x(\frac{1-\beta t}{\beta})} dx = \left(\frac{1}{\beta^\alpha}\right) \left(\frac{\beta}{1-\beta t}\right)^\alpha \int_0^{\infty} \frac{1}{\Gamma(\alpha)\left(\frac{\beta}{1-\beta t}\right)^\alpha} x^{\alpha-1} e^{-x(\frac{1-\beta t}{\beta})} dx = \left(\frac{1}{\beta^\alpha}\right) \left(\frac{\beta}{1-\beta t}\right)^\alpha$$

$$M(t) = \frac{1}{(1-\beta t)^\alpha} \text{ or } (1-\beta t)^{-\alpha} \text{ where } (1-\beta t) > 0 \Rightarrow t < \frac{1}{\beta}.$$

Mean and Variance of the Gamma function

$$M'(t) = -\alpha(1-\beta t)^{-\alpha-1}(-\beta) \text{ and } M'(0) = -\alpha(1-\beta \cdot 0)^{-\alpha}(-\beta) = \alpha\beta \Rightarrow \mu = \alpha\beta.$$

$$M''(t) = \alpha\beta \left[-(\alpha+1)(1-\beta t)^{-\alpha-2}(-\beta) \right] \text{ and}$$

$$M''(0) = \alpha\beta \left[-(\alpha+1)(1-\beta \cdot 0)^{-\alpha-2}(-\beta) \right] = \alpha(\alpha+1)\beta^2.$$

$$\sigma^2 = M''(0) - (M'(0))^2 = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2 \Rightarrow \sigma^2 = \alpha\beta^2.$$

Chi-Square Distribution

$$\text{Now, let } \alpha = \frac{r}{2} \text{ where } r \text{ is a positive integer and } \beta = 2. f(x) = \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{2}}; 0 < x < \infty.$$

$$M(t) = \frac{1}{(1-2t)^{\frac{r}{2}}} \text{ or } (1-2t)^{-\frac{r}{2}} \text{ where } t < \frac{1}{2}.$$

X has a Chi-square distribution with r degrees of freedom and it's denoted by $X \sim \chi_{(r)}^2$.

$$\mu = \left(\frac{r}{2}\right)2 = r \text{ and } \sigma^2 = \left(\frac{r}{2}\right)2^2 = 2r$$

Examples 3.3.5 and 3.3.6 on pp. 153-154.

Example 3.3.6. Consider the Gamma function where $X \sim \text{Gamma}\left(\frac{r}{2}, \beta\right)$;

$$f(x) = \frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{\beta}} dx; 0 < x < \infty. \text{ If } Y = \frac{2X}{\beta}, \text{ find the pdf of } Y.$$

$$\text{Start from } G(Y) = P(Y \leq y) = P\left(\frac{2X}{\beta} \leq y\right) = P\left(X \leq \frac{\beta y}{2}\right) = \int_0^{\frac{\beta y}{2}} \frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{\beta}} dx$$

$$g(y) = G'(Y) = \frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}} \left(\frac{\beta y}{2}\right)^{\frac{r}{2}-1} e^{-\frac{\beta y}{2} \cdot \frac{1}{\beta}} \cdot \frac{d}{dy}\left(\frac{\beta y}{2}\right) - 0 = \frac{1}{\Gamma\left(\frac{r}{2}\right)\beta^{\frac{r}{2}}} \left(\frac{\beta y}{2}\right)^{\frac{r}{2}-1} e^{-\frac{\beta y}{2} \cdot \frac{1}{\beta}} \cdot \left(\frac{\beta}{2}\right)$$

$$g(y) = \frac{1}{\Gamma\left(\frac{r}{2}\right)2^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{2}}; 0 < y < \infty. \text{ Hence, } X \sim \chi_{(r)}^2$$

Homework: 3.3.1, 3.3.2, 3.3.3, 3.3.6, 3.3.7, 3.3.16, 3.3.17 pp. 157-158

Note: Problem 3.3.16 should say “Find the cdf of $Y = -2\log X$.”