

## Chapter 3 Some Special Distributions

### Section 3.2 The Poisson Distribution

Consider the series  $1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \frac{m^4}{4!} + \dots = \sum_{x=0}^{\infty} \frac{m^x}{x!}$ . The series converges to  $e^m$ , for all values of  $m$ . Now consider the function  $f(x) = \frac{m^x e^{-m}}{x!}$ ;  $x = 0, 1, 2, \dots$ , and zero elsewhere, where  $m > 0$ .

**Question:** Is  $f(x)$  a probability density function? (Yes)

1. Since  $m > 0$ , then  $f(x) \geq 0$  and

2.  $\sum_{x=0}^{\infty} f(x) = 1$ . Since  $\sum_{x=0}^{\infty} \frac{m^x e^{-m}}{x!} = e^m e^{-m} = e^0 = 1$ .

$f(x)$  is called a Poisson Distribution and the random variable  $X$  is discrete.

There are many applications that follow a Poisson distribution. For example:

1. Let the random variable  $X$  be the number of cars in an intersection in a certain interval of time. Given the appropriate  $m$ ,  $X$  will follow a Poisson distribution.
2. At the grocery store. Let the random variable  $X$  be the number of customers served by the express lane at a given interval of time. Given the appropriate  $m$ ,  $X$  will follow a Poisson distribution.

### Derivation of the Poisson Distribution

To make the derivation easier, first make an outline of the major steps, and then fill in the details, for each step later.

**Step 1:** Definition of the terms and listing of assumptions

Let  $g(x, w) = p(x \text{ changes in an interval of length } w)$  and define  $g(0, 0) = 1$ ,  $g(x, 0) = 0$ .

Furthermore, let the symbol  $O(h)$  represent any function such that  $\lim_{h \rightarrow 0} \frac{O(h)}{h} = 0$ , (Thus  $h^2 = O(h)$ ,

$h^3 = O(h)$ ,  $h^2 + h^3 = O(h)$ ,  $O(h) + O(h) = O(h)$ ).

**The Poisson postulates (or assumptions) are:**

- $g(1, h) = \lambda h + O(h)$ ,  $\lambda > 0$ ,  $h > 0$  where  $\lambda$  is a positive constant. (The probability of one change in a short interval of time  $h$  is approximately proportional to the length of the interval)
- $\sum_{x=2}^{\infty} g(x, h) = O(h)$  (The probability of **two or more** changes in the same short interval of time  $h$  is approximately equal to zero)
- The number of changes in any interval is independent of the number of changes in any non-overlapping interval.

**Step 2:** Show that  $g(0, w) = e^{-\lambda w}$  by getting  $\frac{dg(0, w)}{dw}$  from  $g(0, w+h)$ , and  $g(0, w)$ , and the definition of the derivative.

Note: Before we show **step 2** we need to know the probability of zero changes in a short interval  $h$ , i.e.  $g(0, h)$ . Also note that from **step 1**, parts (a) and (b), we know the following:

$$g(1, h) = \lambda h + O(h) \quad \text{and} \quad \sum_{x=2}^{\infty} g(x, h) = O(h) \quad . \quad \text{Hence,} \quad g(1, h) + \sum_{x=2}^{\infty} g(x, h) = \sum_{x=1}^{\infty} g(x, h) .$$

$$\text{Where} \quad \sum_{x=1}^{\infty} g(x, h) = \lambda h + O(h) + O(h) = \lambda h + O(h) .$$

Now note that

$$g(0, h) + \sum_{x=1}^{\infty} g(x, h) = 1 \quad \Rightarrow \quad g(0, h) = 1 - \sum_{x=1}^{\infty} g(x, h) \quad \Rightarrow \quad g(0, h) = 1 - (\lambda h + O(h)) \quad \Rightarrow$$

$$g(0, h) = 1 - \lambda h - O(h)$$

Now, we can show **step 2**.

Step 1 part (c)

$$g(0, w+h) = g(0, w)g(0, h) = g(0, w)(1 - \lambda h - O(h)) = g(0, w) - \lambda h g(0, w) - O(h)g(0, w)$$

$$g(0, w+h) - g(0, w) = -\lambda h g(0, w) - O(h)g(0, w) \quad \text{Now divide both sides by } h .$$

$$\frac{g(0, w+h) - g(0, w)}{h} = -\lambda g(0, w) - \frac{O(h)}{h} g(0, w) \quad \text{Now take the limit as } h \rightarrow 0 .$$

$$\lim_{h \rightarrow 0} \frac{g(0, w+h) - g(0, w)}{h} = \lim_{h \rightarrow 0} (-\lambda g(0, w)) - \lim_{h \rightarrow 0} \left( \frac{O(h)}{h} g(0, w) \right)$$

$$\frac{dg(0,w)}{dw} = -\lambda g(0,w) \Rightarrow \frac{dg(0,w)}{dw} \cdot \frac{1}{g(0,w)} = -\lambda \Rightarrow \frac{d \ln(g(0,w))}{dw} = -\lambda \quad \text{Now}$$

integrate both sides with respect to  $w$  .

Step 1:  $g(0,0)=1$

$$\ln(g(0,w)) = -\lambda w + C \Rightarrow g(0,w) = e^{-\lambda w + C} \quad \text{For } w=0 \text{ we have}$$

$$g(0,0) = e^{-\lambda \cdot 0 + C} \Rightarrow 1 = e^C \Rightarrow C = 0. \quad \text{Hence, } g(0,w) = e^{-\lambda w} .$$

**Step 3:** Show that  $\frac{dg(x,w)}{dw} = -\lambda g(x,w) + \lambda g(x-1,w)$  from  $g(x,w+h)$ , and  $g(x,w)$ ,

and the definition of the derivative.

First note that

From Step 1 (b), all the terms sum to the  $O(h)$

$$\begin{aligned} g(x,w+h) &= g(x,w)g(0,h) + g(x-1,w)g(1,h) + \overbrace{\{g(x-2,w)g(2,h) + \dots + g(0,w)g(x,h)\}}^{O(h)} \\ g(x,w+h) &= g(x,w)g(0,h) + g(x-1,w)g(1,h) + O(h) \\ g(x,w+h) &= g(x,w)[1 - \lambda h - O(h)] + g(x-1,w)[\lambda h + O(h)] + O(h) \\ g(x,w+h) &= g(x,w) - \lambda h g(x,w) - O(h)g(x,w) + \lambda h g(x-1,w) + O(h)g(x-1,w) + O(h) \\ g(x,w+h) - g(x,w) &= -\lambda h g(x,w) - O(h)g(x,w) + \lambda h g(x-1,w) + O(h)g(x-1,w) + O(h) \end{aligned}$$

Now divide both sides by  $h$  .

$$\frac{g(x,w+h) - g(x,w)}{h} = -\lambda g(x,w) - \frac{O(h)}{h} g(x,w) + \lambda g(x-1,w) + \frac{O(h)}{h} g(x-1,w) + \frac{O(h)}{h}$$

Now take the limit as  $h \rightarrow 0$  .

$$\lim_{h \rightarrow 0} \frac{g(x,w+h) - g(x,w)}{h} = -\lim_{h \rightarrow 0} \lambda g(x,w) - \lim_{h \rightarrow 0} \frac{O(h)}{h} g(x,w) + \lim_{h \rightarrow 0} \lambda g(x-1,w) + \lim_{h \rightarrow 0} \frac{O(h)}{h} g(x-1,w) + \lim_{h \rightarrow 0} \frac{O(h)}{h}$$

$$\frac{dg(x,w)}{dw} = -\lambda g(x,w) + \lambda g(x-1,w)$$

**Step 4:** Express  $e^{\lambda w} g(x,w)$  as an integral using the result in **Step 3** .

$$\frac{dg(x,w)}{dw} = -\lambda g(x,w) + \lambda g(x-1,w) \Rightarrow \frac{dg(x,w)}{dw} + \lambda g(x,w) = \lambda g(x-1,w)$$

Now multiply both sides by  $e^{\lambda w}$  .

$$\Rightarrow e^{\lambda w} \frac{dg(x, w)}{dw} + \lambda e^{\lambda w} g(x, w) = \lambda e^{\lambda w} g(x-1, w)$$

$$\Rightarrow \frac{d(e^{\lambda w} g(x, w))}{dw} = \lambda e^{\lambda w} g(x-1, w) \quad \text{Now integrate both sides from } 0 \text{ to } w.$$

$$\Rightarrow e^{\lambda w} g(x, w) = \int_0^w \lambda e^{\lambda w} g(x-1, w) dw.$$

**Step 5:** Show that  $g(x, w) = \frac{(\lambda w)^x e^{-\lambda w}}{x!}$ . Use the result from **Step 4** and mathematical induction to get the final result.

$$\begin{aligned} \text{Let } x=1 \Rightarrow e^{\lambda w} g(1, w) &= \int_0^w \lambda e^{\lambda w} g(1-1, w) dw = \int_0^w \lambda e^{\lambda w} g(0, w) dw = \int_0^w \lambda \cancel{e^{\lambda w}} \cancel{e^{-\lambda w}} dw \\ &\Rightarrow e^{\lambda w} g(1, w) = \lambda w \Rightarrow g(1, w) = \lambda w e^{-\lambda w} \Rightarrow g(1, w) = \frac{(\lambda w) e^{-\lambda w}}{1}. \end{aligned}$$

$$\begin{aligned} \text{Let } x=2 \Rightarrow e^{\lambda w} g(2, w) &= \int_0^w \lambda e^{\lambda w} g(2-1, w) dw = \int_0^w \lambda e^{\lambda w} g(1, w) dw = \int_0^w \lambda \cancel{e^{\lambda w}} \frac{(\lambda w) e^{-\lambda w}}{1} dw \\ &\Rightarrow e^{\lambda w} g(2, w) = \frac{\lambda^2 w^2}{2.1} \Rightarrow g(2, w) = \frac{\lambda^2 w^2 e^{-\lambda w}}{2.1} \Rightarrow g(2, w) = \frac{(\lambda w)^2 e^{-\lambda w}}{2.1}. \end{aligned}$$

$$\begin{aligned} \text{Let } x=3 \Rightarrow e^{\lambda w} g(3, w) &= \int_0^w \lambda e^{\lambda w} g(3-1, w) dw = \int_0^w \lambda e^{\lambda w} g(2, w) dw = \int_0^w \lambda \cancel{e^{\lambda w}} \frac{(\lambda w)^2 e^{-\lambda w}}{2.1} dw \\ &\Rightarrow e^{\lambda w} g(3, w) = \frac{\lambda^3 w^3}{3.2.1} \Rightarrow g(3, w) = \frac{\lambda^3 w^3 e^{-\lambda w}}{3.2.1} \Rightarrow g(3, w) = \frac{(\lambda w)^3 e^{-\lambda w}}{3!}. \end{aligned}$$

The formula is correct for  $x = 0, 1, 2, 3, \dots$ . Now assume that it holds for  $x-1$ , so that

$$g(x-1, w) = \frac{(\lambda w)^{x-1} e^{-\lambda w}}{(x-1)!}.$$

$$\text{Then } e^{\lambda w} g(x, w) = \int_0^w \lambda e^{\lambda w} g(x-1, w) dw = \int_0^w \lambda \cancel{e^{\lambda w}} \frac{(\lambda w)^{x-1} e^{-\lambda w}}{(x-1)!} dw$$

$$\Rightarrow e^{\lambda w} g(x, w) = \frac{\lambda^x w^x}{x \dots 3.2.1} \Rightarrow g(x, w) = \frac{\lambda^x w^x e^{-\lambda w}}{x \dots 3.2.1} \Rightarrow g(x, w) = \frac{(\lambda w)^x e^{-\lambda w}}{x!}.$$

Since the formula holds for  $x = 2$ , it holds for  $x = 3$ . Since it holds for  $x = 3$ , it holds for  $x = 4$ , e.t.c., for all integer  $x$ .

The Moment-generating function.

$$M(t) = \sum_{x=0}^{\infty} e^{tx} f(x) = \sum_{x=0}^{\infty} e^{tx} \frac{m^x e^{-m}}{x!} \quad \text{where } m = \lambda w$$

$$M(t) = e^{-m} \sum_{x=0}^{\infty} \frac{(me^t)^x}{x!} = e^{-m} e^{me^t} \Rightarrow M(t) = e^{m(e^t-1)} \quad \text{for real values of } t.$$

$$M'(t) = e^{m(e^t-1)} (me^t) \quad \text{and} \quad M''(t) = e^{m(e^t-1)} (me^t) + e^{m(e^t-1)} (me^t)^2.$$

Work for the derivative,  $M'(t)$ . Let  $y = e^{me^t-m}$  and  $u = me^t - m$  then  $y = e^u$ .

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = (e^u) \cdot (me^t) = (e^{me^t-m}) (me^t) = e^{m(e^t-1)} (me^t).$$

$$\text{Now: } \mu = M'(0) = m \quad \text{and} \quad \sigma^2 = M''(0) - (M'(0))^2 = m + m^2 - m^2 = m.$$

Note:  $\mu = \sigma^2 = m > 0$ . Hence,  $f(x) = \frac{\mu^x e^{-\mu}}{x!}$ ;  $x = 0, 1, 2, 3, \dots$ , and zero elsewhere.

Also, the Poisson random variable  $X$  is denoted by  $X \sim \text{Poisson}(\mu)$ .

$$\text{Note: } M(t) = e^{\mu(e^t-1)}$$

Examples 1, 2, and 3 on pages 145-146.

**HW: Learn the derivation of the Poisson distribution.**