

Chapter 3 Some Special Distributions

Section 3.1 The Binomial, Trinomial, and Multinomial Distributions

Binomial: If n is a positive integer, recall that, $(a + b)^n = \sum_{x=0}^n \binom{n}{x} b^x a^{n-x}$.

Now, consider the function $f(x) = \binom{n}{x} p^x (1-p)^{n-x}$; $x = 0, 1, 2, \dots, n$; zero elsewhere where n is a positive integer and $0 < p < 1$.

Question: Is $f(x)$ a probability density function (pdf) ?

1. $f(x) \geq 0$

2. $\sum_{x=0}^n f(x) = 1$. Since $\sum_{x=0}^n f(x) = \sum_{x=0}^n \binom{n}{x} p^x (1-p)^{n-x} = (p + (1-p))^n = (1)^n = 1$.

Yes, $f(x)$ is a pdf and \mathbf{X} is a binomial random variable. Notation: $X \sim b(n, p)$ or $X \sim B(n, p)$ where n and p are the parameters of the distribution.

Conditions for a binomial random experiment; i.e $X \sim b(n, p)$.

1. The experiment is repeated n independent, identical trials.
2. There are two outcomes possible on each trial -- success or failure.
3. The probability of success, p , remains the same from trial to trial.

At each trial,

let the random variable $X_i = \begin{cases} 1 & ; \text{if a success occurs} \\ 0 & ; \text{if a failure occurs} \end{cases}$ and $P(X_i) = \begin{cases} p & ; X_i = 1 \\ 1-p & ; X_i = 0 \end{cases}$.

The experiment is repeated n times, therefore we have n independent random variables; i.e. X_i ; where $i = 0, 1, 2, \dots, n$. That is, $X_1, X_2, X_3, \dots, X_n$.

Let $Y = X_1 + X_2 + X_3 + \dots + X_n$. Then Y is the number of successes in n trials.

Question: What is the probability distribution of Y ? i.e. $P(Y = y) = ?$

We know that y of the variables have the value **one** and $n-y$ have the value **zero**.

1. How many different ways y random variables out of $X_1, X_2, X_3, \dots, X_n$ can have the value

one? Answer: $\binom{n}{y}$.

2. What is the probability in each way?

Answer: We know that $X_1, X_2, X_3, \dots, X_n$ are independent. We also know that y of the variables have the value **one** and $n-y$ have the value **zero**. That is, the probability in each way is $p^y (1-p)^{n-y}$.

Finally, the probability of y is: $P(Y=y) = f(y) = \binom{n}{y} p^y (1-p)^{n-y}; y = 0, 1, 2, \dots, n$.

The mgf for the binomial random variable is:

$$M(t) = \sum_{x=0}^n e^{tx} f(x) = \sum_{x=0}^n e^{tx} \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^n \binom{n}{x} (pe^t)^x (1-p)^{n-x} = ((1-p) + pe^t)^n.$$

$$M(t) = ((1-p) + pe^t)^n$$

$$\mu = E(x) = M'(t) = n((1-p) + pe^t)^{n-1} pe^t \Big|_{t=0} = n(1-p+p)^{n-1} p = np.$$

$$\begin{aligned} E(x^2) &= M''(t) = n(n-1)((1-p) + pe^t)^{n-2} (pe^t)^2 + n((1-p) + pe^t)^{n-1} pe^t \Big|_{t=0} \\ &= n(n-1)p^2 + np = (np)^2 - np^2 + np. \end{aligned}$$

$$\sigma^2 = E(x^2) - (E(x))^2 = (np)^2 - np^2 + np - (np)^2 = np - np^2 = np(1-p).$$

Example 1 : If $X \sim b(4, \frac{1}{9})$, what is the mgf? $M(t) = ((1 - \frac{1}{9}) + \frac{1}{9}e^t)^4 = (\frac{8}{9} + \frac{1}{9}e^t)^4$.

Example 2 : Given the mgf $M(t) = (\frac{1}{6} + \frac{5}{6}e^t)^{10}$, what is the pdf of \mathbf{X} ? $X \sim b(10, \frac{5}{6})$.

The Negative Binomial Distribution:

The binomial says: What is the number of successes in n trials? 0, or 1, or 2, ..., or n ? Here the number of successes is given by the random variable \mathbf{X} .

Now, consider a sequence of trials. Let the random variable \mathbf{Y} denote the total number of failures before the r^{th} success; that is, $\mathbf{Y} + r$ is equal to the total number of trials necessary to produce exactly r successes. Note: r is a fixed positive integer.

Let $Y = 0, 1, 2, 3, \dots$. What is the pdf of \mathbf{Y} ?

Consider the case of obtaining $\mathbf{r} - 1$ successes in the first $\mathbf{Y} + (\mathbf{r} - 1)$ trials and the \mathbf{r}^{th} success on the $\mathbf{Y} + \mathbf{r}$ trial.

There are $\binom{y + r - 1}{r - 1}$ different ways of obtaining the $\mathbf{r} - 1$ successes out of $\mathbf{Y} + (\mathbf{r} - 1)$ trials.

The probability in each way is $p^{r-1}(1-p)^{y+(r-1)-(r-1)} = p^{r-1}(1-p)^y$, that

is, $\binom{y + r - 1}{r - 1} p^{r-1}(1-p)^y$.

Now, the probability of success on the $\mathbf{Y} + \mathbf{r}$ trial is p .

That is $P(Y = y) = \binom{y + r - 1}{r - 1} p^{r-1}(1-p)^y p = \binom{y + r - 1}{r - 1} p^r(1-p)^y$.

The probability distribution for the negative binomial random variable is:

$P(Y = y) = \binom{y + r - 1}{r - 1} p^r(1-p)^y$; $y = 0, 1, 2, 3, \dots$; zero elsewhere.

Notation: $Y \sim \text{Negb}(Y + r, p)$

The mgf for the negative binomial is:

$M(t) = p^r (1 - (1-p)e^t)^{-r}$; where $1 - (1-p)e^t > 0 \Rightarrow t < -\ln(1-p)$.

Geometric distribution: In the case of the negative binomial distribution,

$P(Y = y) = \binom{y + r - 1}{r - 1} p^r(1-p)^y$; $y = 0, 1, 2, 3, \dots$; zero elsewhere, if $\mathbf{r} = 1$ then

$g(y) = \binom{y + 1 - 1}{1 - 1} p^1(1-p)^y = \binom{y}{0} p(1-p)^y = p(1-p)^y$; $y = 0, 1, 2, 3, \dots$; zero elsewhere.

$g(y) = p(1-p)^y$; $y = 0, 1, 2, 3, \dots$ is the geometric distribution with mgf

$M(t) = p (1 - (1-p)e^t)^{-1}$; where $1 - (1-p)e^t > 0 \Rightarrow t < -\ln(1-p)$.

Also note that the negative binomial can be presented in another way. Currently, the random variable \mathbf{Y} is the number of failures necessary to produce \mathbf{r} Successes. Now let the random variable \mathbf{X} be the total number of trials necessary to produce \mathbf{r} Successes. Note: $\mathbf{X} = \mathbf{Y} + \mathbf{r}$. The probability distribution of \mathbf{X} is

$f(x) = \binom{x - 1}{r - 1} p^r(1-p)^{x-r}$; $x = r, r + 1, r + 2, r + 3, \dots$; zero elsewhere.

Question: What is the mean and variance of \mathbf{X} ? Since X is a linear combination of Y , we don't have to do much work. $E(X) = E(Y + r) = E(Y) + r = \frac{r(1-p)}{p} + r = \frac{r-rp+rp}{p} = \frac{r}{p}$. The variance is $Var(X) = Var(Y + r) = Var(Y) + 0 = \frac{r(1-p)}{p^2}$.

The Trinomial Distribution:

If n is a positive integer and $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are fixed constants, we have

$$\sum_{x=0}^n \sum_{y=0}^{n-x} \frac{n!}{x!y!(n-x-y)!} a_1^x a_2^y a_3^{n-x-y}. \text{ Now introduce } \frac{(n-x)!}{(n-x)!}. \text{ That is}$$

$$\sum_{x=0}^n \frac{n!}{x!(n-x)!} a_1^x \sum_{y=0}^{n-x} \frac{(n-x)!}{y!(n-x-y)!} a_2^y a_3^{n-x-y} = \sum_{y=0}^{n-x} \binom{n-x}{y} a_2^y a_3^{n-x-y}$$

$$= \sum_{x=0}^n \frac{n!}{x!(n-x)!} a_1^x (a_2 + a_3)^{n-x} = \sum_{x=0}^n \binom{n}{x} a_1^x (a_2 + a_3)^{n-x} = (a_1 + a_2 + a_3)^n.$$

Consider the function, $f(x, y) = \frac{n!}{x!y!(n-x-y)!} p_1^x p_2^y p_3^{n-x-y}$; zero elsewhere, where \mathbf{x} and \mathbf{y} are nonnegative integers with $x + y \leq n$ and $p_1 + p_2 + p_3 = 1$.

Question: Is $f(x, y)$ a probability density function (pdf)? **Yes**

1. $f(x, y) \geq 0$
2. $\sum_{x=0}^n \sum_{y=0}^{n-x} f(x, y) = (p_1 + p_2 + p_3)^n = (1)^n = 1$.

Moment generating function for the trinomial distribution.

$$M(t_1, t_2) = E(e^{x t_1 + y t_2}) = \sum_{x=0}^n \sum_{y=0}^{n-x} e^{x t_1 + y t_2} \frac{n!}{x!y!(n-x-y)!} p_1^x p_2^y p_3^{n-x-y}$$

$$= \sum_{x=0}^n \sum_{y=0}^{n-x} \frac{n!}{x!y!(n-x-y)!} (p_1 e^{t_1})^x (p_2 e^{t_2})^y p_3^{n-x-y} = (p_1 e^{t_1} + p_2 e^{t_2} + p_3)^n.$$

$$M(t_1, 0) = (p_1 e^{t_1} + p_2 e^0 + p_3)^n = (p_2 + p_3 + p_1 e^{t_1})^n \quad \text{Since}$$

$$p_1 + p_2 + p_3 = 1 \Rightarrow p_2 + p_3 = 1 - p_1 \quad \text{then } (1 - p_1 + p_1 e^{t_1})^n. \quad \text{Note } X \sim b(n, p_1).$$

$$\text{Similarly, } M(0, t_2) = (1 - p_2 + p_2 e^{t_2})^n. \quad \text{Note } Y \sim b(n, p_2).$$

Further more, \mathbf{X} and \mathbf{Y} are not independent since $M(t_1, t_2) \neq M(t_1, 0)M(0, t_2)$.

Since \mathbf{X} and \mathbf{Y} are not independent, we would like to know the conditional pdfs of $f(y|x)$ and $f(x|y)$.

$$\begin{aligned} f(y|x) &= \frac{f(x,y)}{f(x)} = \frac{\frac{n!}{x!y!(n-x-y)!} p_1^x p_2^y p_3^{n-x-y}}{\frac{n!}{x!(n-x)!} p_1^x (1-p_1)^{n-x}} = \frac{\cancel{x!} y! (n-x-y)! \cancel{p_1^x} p_2^y p_3^{n-x-y}}{\cancel{x!} (n-x)! \cancel{p_1^x} (1-p_1)^{n-x}} = \frac{(n-x)!}{y!(n-x-y)!} \frac{p_2^y p_3^{n-x-y}}{(1-p_1)^{n-x}} \\ &= \binom{n-x}{y} \frac{p_2^y p_3^{n-x-y}}{(1-p_1)^{n-x}} \frac{(1-p_1)^{-y}}{(1-p_1)^{-y}} = \binom{n-x}{y} \frac{p_2^y p_3^{n-x-y}}{(1-p_1)^{n-x}} \frac{(1-p_1)^{-y}}{(1-p_1)^{-y}} = \binom{n-x}{y} \left(\frac{p_2}{1-p_1} \right)^y \left(\frac{p_3}{1-p_1} \right)^{n-x-y}. \end{aligned}$$

$$f(y|x) = \binom{n-x}{y} \left(\frac{p_2}{1-p_1} \right)^y \left(\frac{p_3}{1-p_1} \right)^{n-x-y}; \quad y = 0, 1, 2, \dots, n-x. \quad \text{Note: } Y|X \sim b\left(n-x, \frac{p_2}{1-p_1}\right).$$

$$f(x|y) = \binom{n-y}{x} \left(\frac{p_1}{1-p_2} \right)^x \left(\frac{p_3}{1-p_2} \right)^{n-y-x}; \quad x = 0, 1, 2, \dots, n-y. \quad \text{Note: } X|Y \sim b\left(n-y, \frac{p_1}{1-p_2}\right).$$

Since \mathbf{X} and \mathbf{Y} are dependent random variables compute the correlation, $\rho_{1,2}$.

We know the conditional means, since the conditional distributions are binomial random variables.

$$E(Y|X) = (n-x) \left(\frac{p_2}{1-p_1} \right) \quad \text{and} \quad E(X|Y) = (n-y) \left(\frac{p_1}{1-p_2} \right)$$

$$\rho_{1,2}^2 = \left(-\frac{p_2}{1-p_1} \right) \left(-\frac{p_1}{1-p_2} \right) = \frac{p_1 p_2}{(1-p_1)(1-p_2)} \quad \text{and} \quad \rho_{1,2} = -\sqrt{\frac{p_1 p_2}{(1-p_1)(1-p_2)}}$$

Homework: Section 3.1 – 1, 3, 15, 16, 17 pp. 140-142