

## Chapter 2 Multivariate Distributions

### Section 2.5 Independent Random Variables

Joint Density:  $f(X_1, X_2)$  . Marginal pdfs:  $f_1(X_1)$  and  $f_2(X_2)$  .

$$f(X_1, X_2) = \begin{cases} f(X_2|X_1)f_1(X_1) \\ or \\ f(X_1|X_2)f_2(X_2) \end{cases}$$

Now suppose that  $f(X_2|X_1)$  does not depend on  $X_1$  .

Consider  $f_2(X_2) = \int_{-\infty}^{\infty} f(X_1, X_2) dx_1 = \int_{-\infty}^{\infty} f(X_2|X_1)f_1(X_1)dx_1$  so,

$$f_2(X_2) = f(X_2|X_1) \int_{-\infty}^{\infty} f_1(X_1) dx_1 = f(X_2|X_1) .$$

Hence, if  $f(X_1, X_2) = f(X_2|X_1)f_1(X_1)$  and  $f_2(X_2) = f(X_2|X_1)$  then  
 $f(X_1, X_2) = f_1(X_1)f_2(X_2)$  .

Note: The same discussion applies to the discrete random variables.

**Definition (Independence).** Let the random variables  $X_1$  and  $X_2$  have the joint pdf  $f(X_1, X_2)$  and the marginal pdfs  $f_1(X_1)$  and  $f_2(X_2)$ , respectively. The random variables  $X_1$  and  $X_2$  are said to be independent if, and only if,  
 $f(X_1, X_2) = f_1(X_1)f_2(X_2)$  . Otherwise they are said to be dependent.

Note: There may be certain points  $(x_1, x_2) \in \mathcal{A}$  at which  $f(X_1, X_2) \neq f_1(X_1)f_2(X_2)$  . However, if  $A$  is the set of points  $(x_1, x_2)$  at which the equality does not hold, then  $P(A) = 0$  .

**Example 1:**  $f(X_1, X_2) = e^{-x_1 - x_2}$ ;  $0 < x_1 < \infty$ ,  $0 < x_2 < \infty$  .

$f(X_1, X_2) = e^{-x_1} \cdot e^{-x_2} = f_1(X_1)f_2(X_2)$  , where  $f_1(X_1) = e^{-x_1}$ ;  $0 < x_1 < \infty$   
 $f_2(X_2) = e^{-x_2}$ ;  $0 < x_2 < \infty$  .

**Example 2:**  $f(X_1, X_2) = x_1 + x_2$ ;  $0 < x_1 < 1$ ;  $0 < x_2 < 1$ .

$$f(X_1, X_2) = x_1 + x_2 \neq (x_1 + \frac{1}{2})(x_2 + \frac{1}{2}) = f_1(X_1)f_2(X_2), \text{ where } \begin{matrix} f_1(X_1) = x_1 + \frac{1}{2}; & 0 < x_1 < 1 \\ f_2(X_2) = x_2 + \frac{1}{2}; & 0 < x_2 < 1 \end{matrix} .$$

**Theorem 2.5.1:** Let the random variables  $X_1$  and  $X_2$  have the joint pdf  $f(X_1, X_2)$ . Then  $X_1$  and  $X_2$  are independent if and only if  $f(X_1, X_2)$  can be written as a product of a non-negative function of  $x_1$  alone and a product of a non-negative function of  $x_2$  alone. That is,  $f(X_1, X_2) = g(X_1)h(X_2)$ , where  $g(X_1) > 0$ ;  $x_1 \in \mathcal{A}_1$ , and zero elsewhere, and  $h(X_2) > 0$ ;  $x_2 \in \mathcal{A}_2$ , and zero elsewhere.

**Proof:**

$\Rightarrow$  If  $X_1$  and  $X_2$  are independent, then

$$f(X_1, X_2) = f_1(X_1)f_2(X_2). \text{ Thus, } f(X_1, X_2) = g(X_1)h(X_2).$$

$\Leftarrow$  If  $f(X_1, X_2) = g(X_1)h(X_2)$  then

$$f_1(X_1) = \int_{-\infty}^{\infty} f(X_1, X_2) dx_2 = \int_{-\infty}^{\infty} g(x_1)h(x_2) dx_2 = c_1g(x_1)$$

$$f_2(X_2) = \int_{-\infty}^{\infty} f(X_1, X_2) dx_1 = \int_{-\infty}^{\infty} g(x_1)h(x_2) dx_1 = c_2h(x_2).$$

We already know that  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(X_1, X_2) dx_1 dx_2 = 1$ . Then,

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(X_1, X_2) dx_1 dx_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_1)h(x_2) dx_1 dx_2 = \left[ \int_{-\infty}^{\infty} g(x_1) dx_1 \right] \left[ \int_{-\infty}^{\infty} h(x_2) dx_2 \right] = c_1c_2.$$

So,  $f(X_1, X_2) = g(X_1)h(X_2) = c_1g(X_1).c_2h(X_2) = f_1(X_1)f_2(X_2)$ .

Hence,  $X_1$  and  $X_2$  are independent.

**Example 1:**  $f(X_1, X_2) = e^{-x_1-x_2}$ ;  $0 < x_1 < \infty$ ,  $0 < x_2 < \infty$ .

$$f(X_1, X_2) = e^{-x_1} \cdot e^{-x_2} = f_1(X_1)f_2(X_2), \text{ where } \begin{matrix} f_1(X_1) = e^{-x_1}; & 0 < x_1 < \infty \\ f_2(X_2) = e^{-x_2}; & 0 < x_2 < \infty \end{matrix} .$$

$X_1$  and  $X_2$  are independent.

**In Example 2:**  $f(X_1, X_2) = x_1 + x_2$ ;  $0 < x_1 < 1$  ;  $0 < x_2 < 1$  .

$f(X_1, X_2) = x_1 + x_2 \neq (x_1 + \frac{1}{2})(x_2 + \frac{1}{2}) = f_1(X_1)f_2(X_2)$  , where  $f_1(X_1) = x_1 + \frac{1}{2}$ ;  $0 < x_1 < 1$   
 $f_2(X_2) = x_2 + \frac{1}{2}$ ;  $0 < x_2 < 1$  .

$X_1$  and  $X_2$  are dependent.

**Example 3:**  $f(X_1, X_2) = e^{-x_1 - x_2}$ ;  $0 < x_1 < x_2 < \infty$  .

$f(X_1, X_2) = e^{-x_1} \cdot e^{-x_2} = f_1(X_1)f_2(X_2)$  , where  $f_1(X_1) = e^{-x_1}$ ;  $0 < x_1 < \infty$   
 $f_2(X_2) = e^{-x_2}$ ;  $0 < x_2 < \infty$  .

$X_1$  and  $X_2$  are dependent since the sample space is not a product space.

If the sample space is not a product space, that is, it's bounded by a curve that is neither horizontal nor a vertical line, then the random variables  $X_1$  and  $X_2$  are dependent.

**Theorem 2.5.3:** If  $X_1$  and  $X_2$  are independent with marginal pdfs  $f_1(X_1)$  and  $f_2(X_2)$ , respectively, then  $P(a < X_1 < b, c < X_2 < d) = P(a < X_1 < b) \cdot P(c < X_2 < d)$  for every  $a < b$  and  $c < d$ , where  $a, b, c$ , and  $d$  are constants.

**Proof:**

Since  $X_1$  and  $X_2$  are independent, then  $f(X_1, X_2) = f_1(X_1)f_2(X_2)$  .

$$\begin{aligned} P(a < X_1 < b, c < X_2 < d) &= \int_c^d \int_a^b f(X_1, X_2) dx_1 dx_2 = \int_c^d \int_a^b f_1(x_1) f_2(x_2) dx_1 dx_2 \\ &= \int_a^b f_1(x_1) dx_1 \int_c^d f_2(x_2) dx_2 = P(a < X_1 < b) \cdot P(c < X_2 < d). \end{aligned}$$

**Theorem 2.5.4:** Suppose  $X_1$  and  $X_2$  are independent random variables with marginal pdfs  $f_1(X_1)$  and  $f_2(X_2)$ , respectively. Suppose also, that  $E(u(X_1))$  and  $E(v(X_2))$  exist. Then,

$$E(u(X_1) v(X_2)) = E(u(X_1)) \cdot E(v(X_2)).$$

**Proof:**

$$\begin{aligned} E(u(X_1) v(X_2)) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(X_1) v(X_2) f(X_1, X_2) dx_1 dx_2 \\ &= \left[ \int_{-\infty}^{\infty} u(X_1) f_1(X_1) dx_1 \right] \left[ \int_{-\infty}^{\infty} v(X_2) f_2(X_2) dx_2 \right] = E(u(X_1)) \cdot E(v(X_2)). \end{aligned}$$

**Example 4:**  $Cov(X, Y) = E(XY) - \mu_1\mu_2$  .

If  $X$  and  $Y$  are independent then  $E(XY) = E(X)E(Y) = \mu_1\mu_2$ . Hence,

$Cov(X, Y) = E(XY) - \mu_1\mu_2 = E(X)E(Y) - \mu_1\mu_2 = \mu_1\mu_2 - \mu_1\mu_2 = 0$ . Since, the covariance is zero the  $\rho = \frac{cov(X, Y)}{\sqrt{\sigma_1^2\sigma_2^2}} = \frac{0}{\sqrt{\sigma_1^2\sigma_2^2}} = 0$ .

**Theorem 2.5.5:** Suppose the joint mgf,  $M(t_1, t_2)$ , exists for the random variables  $X_1$  and  $X_2$ . Then  $X_1$  and  $X_2$  are independent if and only if  $M(t_1, t_2) = M(t_1, 0)M(0, t_2)$ , that is, the joint mgf factors into the product of the marginal mgfs.

**Proof:**

$\Rightarrow$  If  $X_1$  and  $X_2$  are independent, then

$$M(t_1, t_2) = E(e^{t_1x_1 + t_2x_2}) = E(e^{t_1x_1})E(e^{t_2x_2}) = M(t_1, 0)M(0, t_2) .$$

$\Leftarrow$  Assume the joint mgf of  $X_1$  and  $X_2$  is given by  $M(t_1, t_2) = M(t_1, 0)M(0, t_2)$ .

Note:  $M(t_1, 0) = \int_{-\infty}^{\infty} e^{t_1x_1} f_1(x_1) dx_1$     and     $M(0, t_2) = \int_{-\infty}^{\infty} e^{t_2x_2} f_2(x_2) dx_2$

Thus we have

$$M(t_1, 0)M(0, t_2) = \left[ \int_{-\infty}^{\infty} e^{t_1x_1} f_1(x_1) dx_1 \right] \left[ \int_{-\infty}^{\infty} e^{t_2x_2} f_2(x_2) dx_2 \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1x_1 + t_2x_2} f_1(x_1)f_2(x_2) dx_1 dx_2 .$$

By assumption  $M(t_1, t_2) = M(t_1, 0)M(0, t_2)$ ; so  $M(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1x_1 + t_2x_2} f_1(x_1)f_2(x_2) dx_1 dx_2$ .

But  $M(t_1, t_2)$  is the mgf of  $X_1$  and  $X_2$ . Thus also

$$M(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1x_1 + t_2x_2} f(X_1, X_2) dx_1 dx_2 .$$

Hence, if  $M(t_1, t_2) = M(t_1, 0)M(0, t_2)$ , then  $X_1$  and  $X_2$  are independent.

**HW: 2.5.1 to 2.5.6 pp. 114**