

## Chapters 8 and 9: Sections 8.1, 9.1, and 9.2--Confidence Intervals

### Section 8.1 – The Sampling Distribution of the Mean

**THEOREM** :  $\mu_{\bar{X}} = E(\bar{X}) = \mu$  and  $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$

Note:  $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$  is also called the **standard error of the mean**.

**Case 1: Shape of the Distribution of  $\bar{X}$  if X is Normal i.e;  $X \sim N(\mu, \sigma)$  .**

If a random variable X is normally distributed, then the sampling distribution of the sample mean,  $\bar{X}$ , is also Normally distributed, i.e;

$$\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right).$$

**Case 2: (The Central Limit Theorem-CLT)**

Regardless of the shape of the underlying population, sampling distribution of the sample mean,  $\bar{X}$ , becomes approximately normal as the sample size, **n**, increases. distributed, regardless of the distribution of X. The approximation becomes better and better with increasing sample size. **Note:** A sample size of **n=30** or more is generally considered large.

**Example 1:** Weights of men are normally distributed with a mean of 170 lbs and a standard deviation of 20 lbs. A sample of 16 men is taken and their weights were observed (a) What is the distribution of  $\bar{X}$ , the sample average? (b) Find the probability that a randomly selected man will weigh less than 166 lbs? (c) What is the probability that the average weight for the sample selected will be less than 166 lbs? (d) What is the standard error of the mean? Ans. (a)  $N(170, 5)$  (b) 0.4207 (c) 0.2119 (d) 5

**Example 2:** One hundred small bolts are packed in a box. These bolts are randomly selected from a population with a mean weight of 1 ounce and a standard deviation of 0.01 ounces. Find the probability that the average weight of the bolts in the box is less than 1.0015 ounces. Ans. (0.9332)

**HW: 11, 12, 19, 24 page 389**

## Section 9.1 Confidence Intervals for the Mean, $\mu$ , when $\sigma$ is Known

### Interval Estimation of a Population mean $\mu$

**Case 1:** If  $\sigma$  is known and  $X \sim N(\mu, \sigma)$  or **Case 2** If  $\sigma$  is known and  $n \geq 30$   
 From Case 1 and Case 2 the sampling distribution of  $\bar{X}$  is Normal ; i.e.  $\bar{X} \sim N(\mu, \frac{\sigma}{\sqrt{n}})$ . We can use it to build a  $(1 - \alpha)\%$  confidence interval for  $\mu$ .

**Confidence Interval:** A confidence interval for estimating  $\mu$  is

$[\bar{X} - Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}, \bar{X} + Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}]$  or  $[\bar{X} \pm Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}]$  where  $1 - \alpha$  is the called the confidence level and  $Z_{\frac{\alpha}{2}}$  is the z-value providing an area of  $\frac{\alpha}{2}$  in the upper tail of the standard normal probability distribution.

In short, we denote a confidence interval (CI) as  $\bar{X} \pm Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$ : **Note:**  $Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$  is the sampling error or (margin of error).

**Example 1:** The U.S. Department of Health, Education, and Welfare collected data for 34 males between the ages of 18 and 24. The sample mean,  $\bar{X}$ , is 69.7 inches. Assume that for this population, the standard deviation is  $\sigma = 4.3$  inches.

a) Compute the 60% confidence interval for  $\mu$  and interpret it.

$$\bar{X} \pm Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} \Rightarrow 69.7 \pm (0.8416) \frac{4.3}{\sqrt{34}} \Rightarrow 69.7 \pm 0.62 \Rightarrow (69.08, 70.32) \text{ (Use TI-83/84)}$$

**Interpretation:** We are 60% confident that the mean height of males between the ages of 18 and 24 is somewhere between 69.08 and 70.32 inches.

b) Compute the 99% confidence interval for  $\mu$  and interpret it.

$$\bar{X} \pm Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} \Rightarrow 69.7 \pm (2.5758) \frac{4.3}{\sqrt{34}} \Rightarrow 69.7 \pm 1.9 \Rightarrow (67.8, 71.6) \text{ (Use TI-83/84)}$$

**Interpretation:** We are 99% confident that the mean height of males between the ages of 18 and 24 is somewhere between 67.8 and 71.6 inches.

c) Compare a and b and draw a conclusion

The 99% confidence interval in b is wider than the 60% confidence interval in a. We can conclude that as the confidence level increases the interval gets wider.

**Example 2:** The test scores in Math1356 are Normally distributed and have a  $\sigma=8.9$  points. The sample mean,  $\bar{X}$ , is found to be 61.3 points.

a) Compute the 86% confidence interval for  $\mu$  and interpret it if the  $n=8$ .  
(56.66, 65.94) (Use TI-83/84)

**Interpretation:** We are 86% confident that the mean test score in Math1356 is somewhere between 56.66 and 65.94 inches.

b) Compute the 86% confidence interval for  $\mu$  and interpret it if the  $n=24$ .  
(58.62, 63.98) (Use TI-83/84)

**Interpretation:** We are 86% confident that the mean test score in Math1356 is somewhere between 58.62 and 63.98 inches.

c) Compare a and b and draw a conclusion  
The confidence interval in a ( $n=8$ ) is wider than the confidence interval in b, ( $n=24$ ). We can conclude that as the sample size increases the interval gets narrower.

### Determining the Required Sample Size

Note that the **sampling error** is:  $E = Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$ . Solving for n, we get

$$n = \left( \frac{Z_{\frac{\alpha}{2}} \sigma}{E} \right)^2 .$$

**Example 4:** Find the sample size necessary to estimate the mean life of a brand of tires if we want the estimate to be within 500 miles of the true mean with 99% confidence. It is known that  $\sigma = 1,200$  miles.

$$\text{Soln. } n = \left( \frac{Z_{\frac{\alpha}{2}} \sigma}{E} \right)^2 = \left( \frac{(2.575)(1,200)}{500} \right)^2 = 38.19 \quad \text{Ans. 39 tires.}$$

**Example 5:** We wish to be 95% sure that a random sample of IQ scores yields a sample mean that is within 2 points of its true mean. How large should the sample be? Assume that  $\sigma = 15$ .

$$\text{Soln. } n = \left( \frac{Z_{\frac{\alpha}{2}} \sigma}{E} \right)^2 = \left( \frac{(1.96)(15)}{2} \right)^2 = 216.09 \quad \text{Ans. 217 scores}$$

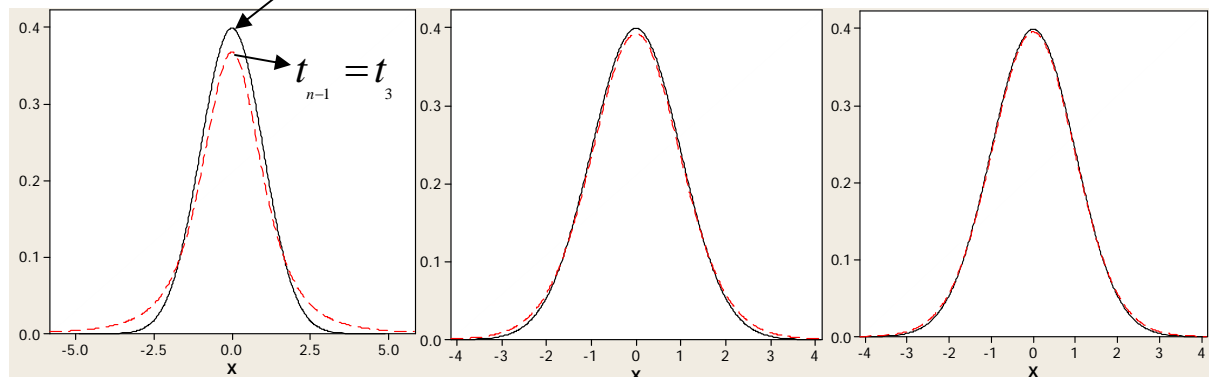
**HW:** 21, 22, 33, 34, 43, 44, 45, 46, 48 pp 416-421 .

## Section 9.2 Confidence Intervals for the Mean when $\sigma$ is Unknown

If the population being sampled has a normal probability distribution with  $\sigma$  unknown and the sample s.d.,  $S$ , is used to estimate  $\sigma$ , statistical inferences about a population mean,  $\mu$ , can be based on the random variable

$$t_{n-1} = \frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}}.$$

The sampling distribution of  $\bar{X}$  is a  $t_{n-1}$ ; i.e.  $\bar{X} \sim t_{n-1}$ . It's called the  $t$  distribution with  $n-1$  degrees of freedom. The  $t$  distribution is actually a family of probability distributions. Each specific distribution depends upon a parameter known as the degrees of freedom (df) where  $df = n - 1$ . As the number of degrees of freedom increases, the  $t$ -distribution and the standard normal distribution,  $Z \sim N(0,1)$ , look the same (see the graphs below).



The first(red curve) has  $df=3$ , the second(red curve)  $df=15$ , and the third(red curve)  $df=29$ . All three graphs are compared to the standard Normal distribution,  $Z \sim N(0,1)$ . As the degrees of freedom increases the  $t$ -distribution is approaching the standard Normal distribution.

### Confidence Interval using the $t$ -distribution

A C.I. for  $\mu$  is  $\bar{X} \pm t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$  where  $1 - \alpha$  is the confidence coefficient and  $t_{\frac{\alpha}{2}}$  is the  $t$ -value providing an area of  $\frac{\alpha}{2}$  in the upper tail of a  $t$ -distribution with  $n - 1$  degrees of freedom.

**Note:**  $t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$  is the sampling error or (margin of error).

**Example 6:** A manufacturer wants to estimate the speed of its new printer. A random sample of 24 printers yielded a mean of 111.875 characters per second with a sd of 8 characters per second. Assume that speed of these printers is normally distributed.

- Construct a 95% confidence interval for the population mean.
- Construct a 99% confidence interval for the population mean.
- Is your 99% confidence interval wider or narrower than 95% confidence interval? Explain your answer carefully.

Soln. (a) A 95% C.I. is  $\bar{X} \pm t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}} = (108.496, 115.254)$  Using TI-83/84

(b) A 99% C.I. is  $\bar{X} \pm t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}} = (107.291, 116.459)$  Using TI-83/84

(c) Wider than 95%. The confidence level was increased to 99%.

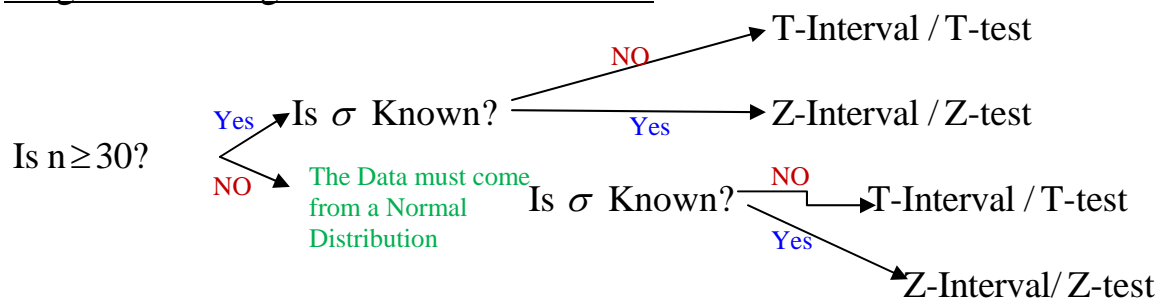
**Example 7:** An economist wants to estimate the average annual income of widows living in a certain city. A random sample of 25 widows gives a sample mean of \$11,700 with a sample standard deviation of \$2,400. Calculate a 90% confidence interval for the average income of the population. You may assume that that annual income for widows has a normal distribution.

Soln. A 90% CI is  $\bar{X} \pm t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}} = (10879, 12521)$  Using TI-83/84

**Example 8:** A sample of 10 cracker boxes yielded the following net weights (in ounces): 16.1, 16.01, 15.82, 15.93, 16.05, 16.05, 15.86, 15.90, 16.05, 16.13. Use the t-distribution to construct a 95% confidence interval for the population mean. You may assume that "net weight" is a normal random variable. Soln.  $\bar{X} = 15.99$  ounces;  $s = 0.1056$

A 95% CI is  $\bar{X} \pm t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}} = (15.915, 16.065)$  Using TI-83/84

Diagram for Using Z-Interval and T-Interval



**HW : 9,10, 11, 12, 13, and 18 pp. 431-432.**